PROBLEM SET 6: Solutions
Math 207A, Fall 2018

1. Plot the bifurcation diagram and representative phases lines for the scalar ODE

$$x_t = \mu x + x^3 - x^5.$$

Identify the bifurcation points and classify them. How would the system behave if μ is increased quasi-statically from $-\infty$ to $+\infty$? When is the equilibrium $x = 0$ linearly stable but unstable to sufficiently large perturbations?

Solution

- The equilibria satisfy

$$x\left(\mu + x^2 - x^4\right) = 0,$$

so $x = 0$ or

$$x^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}.$$

These roots are both complex if $\mu < -1/4$, both real and positive if $-1/4 < \mu < 0$, and one is real and positive the other real an negative of $\mu > 0$. It follows that there are no equilibra in addition to $x = 0$ if $\mu < -1/4$, four additional equilibria if $-1/4 < \mu < 0$, and two additional equilibria if $\mu > 0$.

- A subcritical pitchfork bifurcation occurs at $(x, \mu) = (0, 0)$, and supercritical saddle-node bifurcations occur at $(x, \mu) = \pm(1/2, -1/4)$.

- We have $f_x(0, \mu) = \mu$, so the equilibrium $x = 0$ is asymptotically stable for $\mu < 0$ and unstable if $\mu > 0$. Similarly, one finds that the branches

$$x = \pm\sqrt{\frac{1 + \sqrt{1+4\mu}}{2}}$$

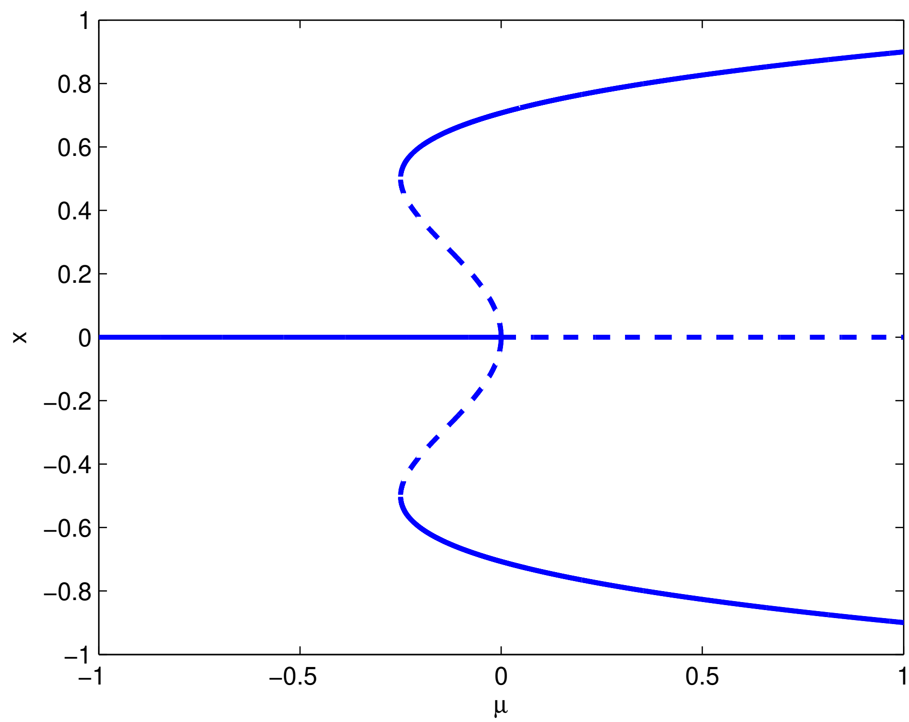
are asymptotically stable if $\mu > -1/4$, and the branches

$$x = \pm\sqrt{\frac{1 - \sqrt{1+4\mu}}{2}}$$

are unstable if $-1/4 < \mu < 0$.

- For $\mu < -1/4$, the equilibrium $x = 0$ is globally asymptotically stable. The system will remain in this state as μ is increased quasistatically to 0. As μ is increased through 0, the system jumps to one of the stable equilibrium branches near $x = \pm 1$. Which one of these symmetric states occurs is not determined by the system, but will depend on external asymmetries, e.g., in the initial data, noise, or imperfections in the system. As μ is increased further, the solution remains on the same stable equilibrium branch.

- As μ is decreased quasistatically from large values, the solution retraces the previous stable equilibrium branch until $\mu = -1/4$ (hysteresis) and then jumps to $x = 0$ for $\mu < -1/4$. The bifurcation diagram is below.



2. Consider the scalar ODE

$$x_t = \lambda + \mu x - x^2,$$

where $\lambda, \mu \in \mathbb{R}$ are parameters. Sketch the bifurcation diagram for the equilibria as a function of μ for fixed λ in the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. Identify the bifurcation points and classify them in each case.

Solution

- The equilibria are

$$\bar{x}(\mu, \lambda) = \frac{-\mu \pm \sqrt{\mu^2 + 4\lambda}}{2}.$$

- For $\lambda = 0$, we get a transcritical bifurcation at $(x, \mu) = 0$. For $\lambda > 0$, this bifurcation is perturbed into two branches (one stable, one unstable) with no bifurcations. For $\lambda < 0$, it is perturbed into two saddle-node bifurcations at

$$(x, \mu) = \left(\bar{x}(\pm 2\sqrt{-\lambda}, \lambda), \pm 2\sqrt{-\lambda}\right).$$

- The stability of these fixed points and a sketch of the bifurcation diagram is left as an exercise.

3. Two rigid rods of length L are connected by a torsional spring with spring constant k and are subject to a compressive force of strength F. Explain why a reasonable model for the potential energy of the system is

$$V(x) = \frac{1}{2}kx^2 - 2FL(1 - \cos x),$$

where x is the angle of the rods to the horizontal. If the rod is strongly damped with damping constant $\beta > 0$, then the ODE for its motion is $\beta x_t + V'(x) = 0$. Show that a nondimensionalized form of the ODE is

$$x_t + x - \mu \sin x = 0, \qquad \mu = \frac{2FL}{k}.$$

Sketch a bifurcation diagram for the ODE and classify the bifurcation that occurs.

Solution

- The potential energy of the spring is

$$V_{\text{spring}}(x) = \frac{1}{2}kx^2$$

  and the work done on the rods by the external force (Force $\times$ Distance) is

$$V_{\text{force}}(x) = 2FL(1 - \cos x).$$

  Then

$$V(x) = V_{\text{spring}}(x) - V_{\text{force}}(x).$$

- We have

$$V'(x) = kx - 2FL \sin x = k\left(x - \mu \sin x\right), \qquad \mu = \frac{2FL}{k}.$$

  The force balance $\beta x_t = -V'(x)$ then gives the ODE, which is nondimensionalized by using the time $\tilde{t} = kt/\beta$. Note that k has dimensions of energy (the angle x is dimensionless) as does FL, so μ is dimensionless.

- The fixed points satisfy $x = \mu \sin x$, and a necessary condition for a bifurcation point is

$$\frac{d}{dx}(x - \mu \sin x) = 1 - \mu \cos x = 0.$$

So an equilibrium bifurcation off the branch $x = 0$ can occur only if $\mu = 1$. In addition, if we restrict our consideration to angles with $|x| < 2\pi$, then $x = 0$.

- Taylor expanding (x, μ) about $(0, 1)$ we write

$$x = x_1 + \ldots, \qquad \mu = 1 + \mu_1 + \ldots.$$

Then

$$\begin{aligned} x - \mu \sin x &= x_1 - (1 + \mu_1) \sin x_1 + \ldots \\ &= x_1 - (1 + \mu_1) \left(x_1 - \frac{1}{6} x_1^3 + \ldots \right) \\ &= -\mu_1 x_1 + \frac{1}{6} x_1^3 + \ldots. \end{aligned}$$

Thus, near the bifurcation point we have to leading order that

$$\mu_1 x_1 = \frac{1}{6} x_1^3$$

corresponding to a supercritical pitchfork bifurcation.

4. Consider the system

$$x_t = 1 - x - \beta xy, \qquad y_t = \beta xy - (1+\gamma)y,$$

where $\beta, \gamma > 0$ are positive parameters. Sketch the bifurcation diagram for the equilibria as a function of β and show that a bifurcation occurs at some $\beta = \beta_*(\gamma)$. What kind of bifurcation is it? Sketch typical phase planes on $\mathbb{R}^2$ (using numerical solutions if you prefer) for β close to β_* when $\beta < \beta_*$, $\beta = \beta_*$, and $\beta > \beta_*$.

Solution

- See the solution to Problem 1 on the midterm.