Problem set 7: Solutions
Math 207A, Fall 2018

1. The normal form for a Takens-Bogdanov bifurcation is the following 2×2 system depending on two parameters (μ, ν):

$$\dot{x} = y, \qquad \dot{y} = \mu + \nu y + x^2 + xy.$$

(a) What is the algebraic structure of the linearized system at $(x, y; \mu, \nu) = (0, 0; 0, 0)$? Find the equilibria, classify them, and determine what bifurcations of equilibria occur as (μ, ν) vary over $(\mu, \nu) \neq (0, 0)$.

(b) Compare your results with the global phase portraits shown in Figure 8.14 of the text by Meiss.

Solution

- When $\mu = \nu = 0$, the matrix in the linearization at $(0, 0)$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

  which is a nondiagonalizable matrix that has a double eigenvalue $\lambda = 0$ with geometric multiplicity one.

- There are no fixed points for $\mu > 0$. The fixed points for $\mu \leq 0$ are

$$(x, y) = (\bar{x}(\mu), 0), \qquad \bar{x}(\mu) = \pm\sqrt{-\mu}.$$

  The Jacobian evaluted at these fixed points is

$$\begin{pmatrix} 0 & 1 \\ 2\bar{x} & \nu + \bar{x} \end{pmatrix}$$

  with eigenvalues

$$\lambda = \frac{1}{2}\left(\nu + \bar{x} \pm \sqrt{(\nu + \bar{x})^2 + 8\bar{x}}\right).$$

- It follows that a saddle node bifurcation occurs at $(x, y; \mu, \nu) = (0, 0; 0, \nu)$ for fixed $\nu \neq 0$, where $\lambda = 0, \nu$ and the two fixed points collide and disappear as μ increases through zero.

- In addition, a Hopf bifurcation can (and does) occur at

$$(x, y; \mu, \nu) = (-\sqrt{-\mu}, 0; \mu, \sqrt{-\mu})$$

for $\mu < 0$ and $\nu > 0$, when $\lambda = \pm i\sqrt{2\sqrt{-\mu}}$ is a purely imaginary complex conjugate pair.

- There is a further global homoclinic bifurcation in which the limit cycle created in the Hopf bifurcation disappears by colliding with a saddle-point. This bifurcation can be shown to occur on a curve

$$\mu = -\frac{49}{25}\nu^2 + O(\nu^{5/2}) \qquad \text{as } \nu \to 0^+.$$

2. (a) Consider a Hamiltonian system for $x, y \in \mathbb{R}^n$

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}$$

with Hamiltonian $H(x, y) \in \mathbb{R}$. Show that if λ is an eigenvalue of the linearization at an equilibrium, then so is $-\lambda$. What are the possible forms of the eigenvalues for a 2×2 system? HINT. Show that the matrix of the linearization has the form JA where A is symmetric and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

(b) Suppose $x, y \in \mathbb{R}$ and

$$H(x, y, \mu) = \frac{1}{2}y^2 + V(x, \mu), \qquad V(x, \mu) = \frac{1}{4}x^4 - \frac{1}{2}\mu x^2,$$

where $\mu \in \mathbb{R}$ is a parameter. Determine the equilibria of the corresponding Hamiltonian system as a function of μ, classify them. Sketch the resulting phase planes and the bifurcation diagram.

Solution

- The linearized system has the form

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = JA \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} \partial^2 H/\partial x^2 & \partial^2 H/\partial x \partial y \\ \partial^2 H/\partial y \partial x & \partial^2 H/\partial y^2 \end{pmatrix}$$

  where the Hessian matrix A of H is evaluated at the fixed point. The equality of mixed partial derivatives of H, which we assume is smooth, implies that A is symmetric.

- Let

$$p(\lambda) = \det (JA - \lambda I)$$

  denote the characteristic polynomial of the Jacobian matrix JA. Then, since a matrix and its transpose have the same determinant, and

$$J^T = -J, \qquad J^2 = -I, \qquad \det J^2 = 1,$$

we have that

$$\begin{aligned} p(\lambda) &= \det (JA - \lambda I)^T \\ &= \det(-AJ - \lambda I) \\ &= \det[-J(AJ + \lambda I) J] \\ &= \det(JA + \lambda I) \\ &= p(-\lambda). \end{aligned}$$

It follows that if λ is a root of the characteristic polynomial, so is $-\lambda$.

- By reality, if $\lambda \in \mathbb{C}$ is an eigenvalue of JA, so is $\bar{\lambda}$. It follows that for a 2×2 system, the only three possibilities are: (i) two distinct real eigenvalues $\{\lambda, -\lambda\}$ where $\lambda > 0$; (ii) a pair of purely imaginary eigenvalues $\{i\omega, -i\omega\}$ where $\omega > 0$; (iii) an eigenvalue $\lambda = 0$ with algebraic multiplicity two.

- The equilibria are given by $H_x = H_y = 0$, which imples that $y = 0$ and

$$x^3 - \mu x = 0,$$

so $x = 0$ or $x = \pm\sqrt{\mu}$ for $\mu > 0$.

- The matrix of the Jaconian matrix at $(x, y) = (\bar{x}, 0)$ is

$$JA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3\bar{x}^2 - \mu & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mu - 3\bar{x}^2 & 0 \end{pmatrix},$$

with eigenvalues

$$\lambda = \pm\sqrt{\mu - 3\bar{x}^2}.$$

- It follows that $(0, 0)$ is a stable center for $\mu < 0$ and an unstable saddle point for $\mu > 0$. The equilibria $(\pm\sqrt{\mu}, 0)$ are stable centers for $\mu > 0$.

- This center-saddle bifurcation is a (supercritical) Hamiltonian pitchfork bifurcation. In it, a center turns into a saddle and two new centers appear, while a complex conjugate pair of eigenvalues collide at the origin and split into positive and negative real eigenvalues. This differs from a standard pitchfork bifurcation in nonconservative systems, where a stable node turns into a saddle point and two new stable nodes appear, while a single real eigenvalue crosses the origin.