PROBLEM SET 8
Math 207A, Fall 2018
Due: Fri., Dec. 7

1. Sketch the phase plane of the system

$$x_t = x^2, \qquad y_t = -y.$$

Linearize the system about the equilibrium $(0,0)$ and determine the unstable, stable, and center subspaces of the equilibrium. What is the stable manifold $W^s(0,0)$? Show that there are many possible choices of an invariant (C^1) center manifold $W^c(0,0)$ that is tangent to the center subspace at $(0,0)$.

2. The following Selkov system for $x(t), y(t) \geq 0$, depending on parameters $a, \mu > 0$, provides a simple model of glycolysis:

$$x_t = -x + ay + x^2 y, \qquad y_t = \mu - ay - x^2 y.$$

(a) Find the fixed point and classify it as a function of the parameters. Show that if $0 < a < 1/8$, then there are possible Hopf bifurcations as μ increases from 0 to ∞. What are the possible Hopf bifurcation points $(x_0, y_0; \mu_0)$?

(b) By plotting the phase planes numerically, show that Hopf bifurcations occur at the points in (a) and determine whether they are subcritical or supercritical.

3. Consider the following discrete dynamical system for $x_n \in \mathbb{R}$ depending on a parameter $\mu \in \mathbb{R}$:

$$x_{n+1} = -\frac{\mu}{2} \tan^{-1} x_n.$$

(a) Describe the fixed point(s) of the system and determine their stability. What bifurcations of fixed points occur as μ increases from $-\infty$ to ∞?

(b) Show that a period-doubling bifurcation occurs at $\mu = 2$. Is the resulting period-two orbit stable or unstable?

4. The Hénon map on $\mathbb{R}^2$ is given by

$$\begin{aligned} x_{n+1} &= a - by_n - x_n^2, \\ y_{n+1} &= x_n. \end{aligned}$$

(a) Find the fixed points and determine their stability.

(b) Carry out a numerical exploration of this map for various values of the parameters $a, b \in \mathbb{R}$. It's up to you how much you want to explore, especially at the end of the quarter, but you should provide a plot of the forward orbit of the point $(x_0, y_0) = (0, 0)$ for $a = 1.4$ and $b = -0.3$ in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ and briefly discuss the result.