PROBLEM SET 8: Solutions
Math 207A, Fall 2018

1. Sketch the phase plane of the system

$$x_t = x^2, \qquad y_t = -y.$$

Linearize the system about the equilibrium $(0,0)$ and determine the unstable, stable, and center subspaces of the equilibrium. What is the stable manifold $W^s(0,0)$? Show that there are many possible choices of an invariant (C^1) center manifold $W^c(0,0)$ that is tangent to the center subspace at $(0,0)$.

Solution

- the linearized system at $(0,0)$ is

  $$x_t = 0, \qquad y_t = -y.$$

  This has eigenvalues $\lambda = -1, 0$ with corresponding eigenvectors $\vec{r} = (0,1)^T, (1,0)^T$ which span the (one-dimensional) stable and center subspaces, respectively. The (zero-dimensional) unstable subspace consists of 0.

- The equation of the trajectories is

  $$\frac{dy}{dx} = -\frac{y}{x^2}$$

  Separating variables and solving this equation, we find that

  $$y = Ce^{1/x}$$

  where C is a constant of integration.

- For $C \neq 0$, the trajectories approach the origin smoothly (C^∞) as $x \to 0^-$, go to infinity as $x \to 0^+$,a and approach the horizontal asymptote $y = C$ as $|x| \to \infty$.

- The stable subspace of the origin, the y-axis, is invariant under the flow, so it is also the stable manifold. The unstable manifold is 0.

- Any curve of the form

$$y = \begin{cases} Ce^{1/x} & -\infty < x < 0 \\ 0 & 0 \le x \end{cases}$$

for some constant C is a smooth (C^∞) invariant manifold. (It consists of three trajectories: the part for $x < 0$, the equilibrium 0, and the positive x-axis.) It is tangent to the center subspace at 0, so any such curve is a center manifold. In particular, the whole x-axis is a center manifold ($C = 0$), but it is not the only one.

2. The following Selkov system for $x(t), y(t) \geq 0$, depending on parameters $a, \mu > 0$, provides a simple model of glycolysis:

$$x_t = -x + ay + x^2 y, \qquad y_t = \mu - ay - x^2 y.$$

(a) Find the fixed point and classify it as a function of the parameters. Show that if $0 < a < 1/8$, then there are possible Hopf bifurcations as μ increases from 0 to ∞. What are the possible Hopf bifurcation points $(x_0, y_0; \mu_0)$?

(b) By plotting the phase planes numerically, show that Hopf bifurcations occur at the points in (a) and determine whether they are subcritical or supercritical.

Solution

- The fixed point is
$$\bar{x} = \mu, \qquad \bar{y} = \frac{\mu}{a + \mu^2}.$$
The fixed point changes stability, with a complex conjugate pair of eigenvalues passing through the imaginary axis, when
$$\mu^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1 - 8a}\right),$$
so these are the possible Hopf bifircation points.

- Numerical solutions show that there is a supercritical Hopf bifurcation as μ increases through the smaller value, and a subcritical Hopf bifurcation as μ increases through the larger value.

Remark. See S. H Strogatz, *Nonlinear dynamics and Chaos*, pp. 205–209 for more details (where $\mu = b$). The Wikipedia page on Hopf Bifurcation has a nice animation of the numerical solutions.

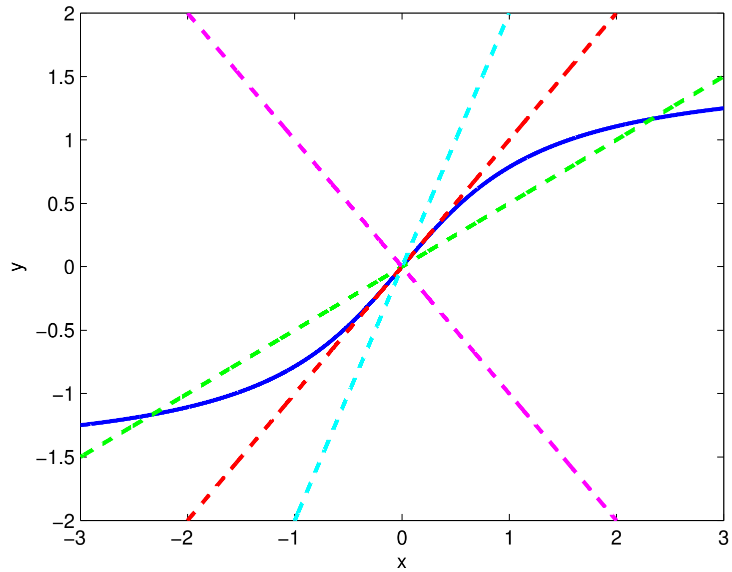

Figure 1: Graphs of $y = \tan^{-1} x$ (blue) and $y = -2x/\mu$ for: $\mu = -4$ (green); $\mu = -2$ (red); $\mu = -1$ (cyan); $\mu = 2$ (magenta).

3. Consider the following discrete dynamical system for $x_n \in \mathbb{R}$ depending on a parameter $\mu \in \mathbb{R}$:

$$x_{n+1} = -\frac{\mu}{2} \tan^{-1} x_n.$$

(a) Describe the fixed point(s) of the system and determine their stability. What bifurcations of fixed points occur as μ increases from $-\infty$ to ∞?

(b) Show that a period-doubling bifurcation occurs at $\mu = 2$. Is the resulting period-two orbit stable or unstable?

Solution

- (a) The fixed points of the map

$$f(x;\mu) = -\frac{\mu}{2} \tan^{-1} x$$

correspond to intersections of the line $\mu y = -2x$ with the graph $y = \tan^{-1} x$. (See Figure 1.)

- If $\mu < -2$, then the slope of the line is positive and less than the slope of the inverse tangent at $x = 0$, and there are three such points with

$$x = 0, \quad x = \pm\bar{x}(\mu)$$

where $\bar{x}(\mu) > 0$ satisfies

$$-\frac{2}{\mu}\bar{x} = \tan^{-1}\bar{x}.$$

If $\mu \geq -2$, then there is a unique fixed point at $x = 0$.

- If $\mu = -2$, then the line is tangent to the graph of the inverse tangent at $x = 0$, and a subcritical pitchfork bifurcation occurs as μ increases through -2. There are no other bifurcations of fixed points.

- We have

$$f_x(x;\mu) = -\frac{\mu}{2}\frac{1}{1+x^2}.$$

Thus, $f_x(0;\mu) = -\mu/2$, so $|f_x(0,\mu)| < 1$ if $|\mu| < 2$ and $|f_x(0,\mu)| > 1$ if $|\mu| > 2$. It follows that the fixed point $x = 0$ is asymptotically stable if $|\mu| < 2$ and unstable if $|\mu| > 2$.

- The fixed point $x = 0$ gains stability as μ increases through -2 and the eigenvalue $f_x(0;\mu)$ decreases through 1 (corresponding to a bifurcation of fixed points); and loses stability as μ increases through 2 and the eigenvalue $f_x(0;\mu)$ decreases through -1.

- If $\mu < -2$, then

$$f_x\left(\pm\bar{x}(\mu);\mu\right) = -\frac{\mu}{2}\frac{1}{1+\bar{x}^2(\mu)}.$$

The graph of $y = \tan x$ has smaller slope than the line $y = -2x/\mu$ at $x = \bar{x}(\mu)$, so $0 < f_x\left(\bar{x}(\mu);\mu\right) < 1$, and these fixed points are stable. This claim is clear geometrically, but we omit an analytical proof.

- (b) A period doubling bifurcation can occur at $(x, mu) = (0, 2)$ since the eigenvalue $f_x(0;\mu)$ decreases through -1. A discussion of whether the bifurcation is subcritical or supercritical and the stability of the periodic orbit is omitted.

4. The Hénon map on $\mathbb{R}^2$ is given by

$$x_{n+1} = a - by_n - x_n^2,$$
$$y_{n+1} = x_n.$$

(a) Find the fixed points and determine their stability.
(b) Carry out a numerical exploration of this map for various values of the parameters $a, b \in \mathbb{R}$. It's up to you how much you want to explore, especially at the end of the quarter, but you should provide a plot of the forward orbit of the point $(x_0, y_0) = (0, 0)$ for $a = 1.4$ and $b = -0.3$ in the region $-2 \le x \le 2$, $-2 \le y \le 2$ and briefly discuss the result.

Solution

- (a) The fixed points (x, y) satisfy

  $$x = a - by - x^2, \qquad y = x$$

  which gives

  $$x = y = \frac{1}{2}\left[-(1+b) \pm \sqrt{(1+b)^2 + 4a}\right].$$

  There are no fixed points if $4a < -(1+b)^2$.

- The Jacobian matrix is

  $$D_{(x,y)}f = \begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix},$$

  with eigenvalues

  $$\lambda = -x \pm \sqrt{x^2 - b}.$$

  The fixed points are asymptotically stable if $|\lambda| < 1$ for both eigenvalues, and unstable if $|\lambda| > 1$ for at least one eigenvalue.

- For $a = 1.4$, $b = -0.3$, we find (hopefully correctly) that $x \approx 0.884$ with eigenvalues $\lambda \approx 0.156, -1.924$, or $x \approx -1.584$ with eigenvalues $\lambda \approx -0.092, 3.260$. Both fixed points are (orientation reversing) saddle points with $|\lambda_1| < 1$ and $|\lambda_2| > 1$.

- (b) Numerical solutions should show that the orbit approaches the Hénon attractor, which is a strange attractor with a fractal, Cantor set structure. This chaotic behavior is associated with a homoclinic tangle that results from transverse intersection of the stable and unstable manifolds of the saddle points.