Final Exam
Math 207B, Winter 2012
Solutions

1. [20 pts] (a) Find the Green's function for the following BVP

$$-u'' = f(x), \qquad 0 < x < 1,$$
$$u(0) = 0, \qquad u'(1) = 0.$$

Draw a graph of the Green's function $G(x, \xi)$ versus x for fixed $0 < \xi < 1$.
(b) Evaluate the Green's function representation of the solution

$$u(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi$$

in the case when $f(x) = x$, and verify that it is the solution of the BVP.

Solution

- (a) Solutions of the homogeneous equation $-u'' = 0$ that satisfy the left and right boundary conditions are $u_1(x) = x$, $u_2(x) = 1$, with Wronskian -1. The Green's function is

$$G(x, \xi) = \begin{cases} x & 0 \le x \le \xi, \\ \xi & \xi \le x \le 1. \end{cases}$$

- The Green's function is linear for $0 \le x \le \xi$ and constant for $\xi \le x \le 1$. Physically, this solution describes (for example) the steady temperature due to a point source in a rod whose left end is held at a fixed temperature and whose right end is insulated. The temperature u is constant between the point source and the insulated end, and there is a constant heat flux $-u_x$ from the point source to the end held at a fixed temperature.

- (b) For $f(x) = x$, the Green's function representation of the solution gives

$$
\begin{aligned}
u(x) &= \int_0^1 G(x,\xi)\xi \, d\xi \\
&= \int_0^x \xi \cdot \xi \, d\xi + \int_x^1 x \cdot \xi \, d\xi \\
&= \left[\frac{1}{3}\xi^3\right]_0^x + x\left[\frac{1}{2}\xi^2\right]_x^1 \\
&= \frac{1}{2}x - \frac{1}{6}x^3.
\end{aligned}
$$

- This solution is the sum of a particular solution $u_p(x) = -\frac{1}{6}x^3$ of the non-homogeneous ODE and a solution $u_h(x) = \frac{1}{2}x$ of the homogeneous ODE which ensures that $u(x)$ satisfies the BCs at $x = 0$, $x = 1$.

2. [25 pts] (a) Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u'(0) &= 0, \quad u'(1) = 0. \end{aligned}$$

(b) If $0 < \xi < 1$, write down the expansion of the delta-function $\delta(x - \xi)$, regarded as a function of x, with respect to the eigenfunctions from part (a).

(c) Use separation of variables to solve the following IBVP for the heat equation for $u(x, t; \xi)$ with an initial point source located at $x = \xi$, where $0 < \xi < 1$,

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1 \quad t > 0, \\ &u_x(0, t; \xi) = 0, \quad u_x(1, t; \xi) = 0, \\ &u(x, 0; \xi) = \delta(x - \xi). \end{aligned}$$

How does the solution behave as $t \to \infty$?

Solution

- (a) The eigenvalues and eigenfunctions are

  $$\lambda_n = n^2\pi^2, \qquad \phi_n(x) = \cos n\pi x$$

  where $n = 0, 1, 2, 3, \ldots$. These are normalized so that

  $$\int_0^1 \phi_0^2 \, dx = 1, \qquad \int_0^1 \phi_n^2 \, dx = \frac{1}{2} \quad \text{for } n = 1, 2, 3, \ldots.$$

- (b) The eigenfunction expansion of a function $f(x)$ is the Fourier cosine series

  $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x, \qquad a_n = 2\int_0^1 f(x) \cos n\pi x \, dx.$$

  For $f(x) = \delta(x - \xi)$, we get $a_n = 2\cos n\pi\xi$, and

  $$\delta(x - \xi) = 1 + 2\sum_{n=1}^{\infty} \cos(n\pi x)\cos(n\pi\xi)$$

  where the series converges in a distributional sense.

- (c) The separated solutions of the heat equation are

$$u(x,t) = X(x)e^{-\lambda t}$$

where

$$-X'' = \lambda X, \qquad X'(0) = 0, \quad X'(1) = 0.$$

It follows that $\lambda = n^2\pi^2$ and $X = \cos n\pi x$, as above, and by superposition the general solution for $u(x,t)$ is

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x).$$

Imposing the initial condition, we find that the coefficients a_n are the ones obtained in (b), so that

$$u(x,t;\xi) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\cos(n\pi x)\cos(n\pi\xi).$$

- As $t \to \infty$ all the terms approach zero except for the constant term, and so

$$u(x,t;\xi) \to 1 \qquad \text{as } t \to \infty.$$

This constant is what we would expect from conservation of energy, since

$$\frac{d}{dt}\int_0^1 u(x,t)\,dx = 0, \qquad \int_0^1 \delta(x-\xi)\,dx = 1.$$

Remark. The solution in (c) is the Green's function for this IBVP. Writing this Green's function as $G(x,t;\xi)$, instead of $u(x,t;\xi)$, we can represent the solution $u(x,t)$ of the IBVP with general initial data

$$\begin{aligned} &u_t = u_{xx}, \\ &u_x(0,t) = 0, \qquad u_x(1,t) = 0, \\ &u(x,0) = f(x) \end{aligned}$$

as

$$u(x,t) = \int_0^1 G(x,t;\xi)f(\xi)\,d\xi.$$

Note that $G(x,t;\xi)$ a smooth function for all $t > 0$ since its Fourier coefficients decay exponentially as $n \to \infty$.

3. [15 pts] Show that the solution $u(x)$ of the Fredholm integral equation

$$u(x) - \int_0^1 (x+y)\, u(y)\, dy = 1, \qquad 0 \le x \le 1$$

has the form $u(x) = ax + b$ for some constants a, b and solve the equation.

Solution

- The integral equation is a degenerate self-adjoint Fredholm equation of the second kind with kernel

$$k(x,y) = x \cdot 1 + 1 \cdot y.$$

- It follows from the equation that

$$u(x) = \left(\int_0^1 u(y)\, dy \right) x + \int_0^1 y u(y)\, dy + 1,$$

so $u(x) = ax + b$ where

$$a = \int_0^1 u(y)\, dy, \qquad b = \int_0^1 y u(y)\, dy + 1.$$

- Using $u = ax + b$ in these equations, we get

$$\begin{aligned} a &= \int_0^1 (ay+b)\, dy = \frac{1}{2}a + b, \\ b &= \int_0^1 y(ay+b)\, dy = \frac{1}{3}a + \frac{1}{2}b + 1 \end{aligned}$$

or

$$a - 2b = 0, \qquad -2a + 3b = 6.$$

- This is a non-singular system (meaning that 1 is not an eigenvalue of the integral operator) with solution $a = -12$, $b = -6$, so the unique solution of the integral equation is

$$u(x) = -6(2x+1).$$

4. [10 pts] Suppose that $u(x)$ minimizes the functional

$$J(u) = \int_0^1 \left\{ \frac{1}{2}[u'(x)]^2 - xu(x) \right\} dx$$

over the space of C^2-functions such that $u(0) = 0$, $u'(1) = 0$. Find the boundary value problem satisfied by u and solve for u.

Solution

- The Euler-Lagrange equation for this functional is

$$-u'' = x, \qquad u(0) = 0, \quad u'(1) = 1.$$

  The solution is the function from Problem 1,

$$u(x) = \frac{1}{2}x - \frac{1}{6}x^3.$$

- To derive the Euler-Lagrange equation, we compute for any smooth function $h(x)$ such that $h(0) = h'(1) = 0$ that

$$\begin{aligned} \frac{d}{d\epsilon} J(u + \epsilon h)\bigg|_{\epsilon=0} &= \frac{d}{d\epsilon} \int_0^1 \left\{ \frac{1}{2}(u' + \epsilon h')^2 - x(u + \epsilon h) \right\} dx \bigg|_{\epsilon=0} \\ &= \int_0^1 (u'h' - xh)\ dx \\ &= [u'h]_0^1 - \int_0^1 (u''h + xh)\ dx \\ &= -\int_0^1 (u'' + x)\, h\, dx. \end{aligned}$$

  If u minimizes J on functions that satisfy the BCs, then this derivative of J in the direction h must vanish for all functions h, which implies that $u'' + x = 0$.

5. [10 pts] Suppose that Ω is a smooth, bounded region in $\mathbb{R}^n$. Use a Green's identity to show that if $\lambda \le 0$, then the only solution of the BVP

$$\begin{aligned} -\Delta u &= \lambda u & x &\in \Omega \\ u &= 0 & x &\in \partial\Omega \end{aligned}$$

is $u = 0$, so λ is not an eigenvalue. Why doesn't the argument apply if $\lambda > 0$?

Solution

- By Green's first identity (or the divergence theorem)

$$\int_\Omega \left(u\Delta u + |\nabla u|^2\right)\, dx = \int_\Omega \nabla \cdot (u\nabla u)\, dx = \int_{\partial\Omega} u\frac{\partial u}{\partial n}\, dS.$$

  The integral over $\partial\Omega$ is zero since $u = 0$ on the boundary. Using the PDE we get

$$\int_\Omega \left(-\lambda u^2 + |\nabla u|^2\right)\, dx = 0.$$

- If $\lambda \le 0$ then all the terms in this equation are nonnegative, so

$$\lambda \int_\Omega u^2\, dx = 0, \qquad \int_\Omega |\nabla u|^2\, dx = 0.$$

  If $\lambda < 0$, we conclude immediately from the first equation that $u = 0$. If $\lambda = 0$, then $\nabla u = 0$, so $u =$ constant, and then $u = 0$ since it vanishes on the boundary.

- If $\lambda > 0$, this argument fails because the integrals may cancel. In fact, if λ is an eigenvalue of the Dirichlet Laplancian with eigenfunction ϕ, then we have

$$\lambda = \frac{\int_\Omega |\nabla \phi|^2\, dx}{\int_\Omega \phi^2\, dx}.$$

  The right-hand side of this equation is the Rayleigh quotient for the Laplacian.

Extra Credit Question: Attempt only if time permits

Consider the Sturm-Liouville equation for $u(x)$

$$-(pu')' + qu = \lambda ru \tag{1}$$

where $p(x)$, $q(x)$, $r(x)$ are given smooth coefficient functions and λ is a constant. Write

$$u = \rho \sin\theta, \qquad pu' = \rho\cos\theta \tag{2}$$

where $\rho(x)$, $\theta(x)$ are two new functions. Show that ρ, θ satisfy the differential equations

$$\begin{aligned} \theta' &= \frac{1}{p}\cos^2\theta + (\lambda r - q)\sin^2\theta, \\ \rho' &= \left(q - \lambda r + \frac{1}{p}\right)(\sin\theta\cos\theta)\,\rho. \end{aligned} \tag{3}$$

Explain why a solution of the initial-value problem for these equations, with

$$\theta(x_0) = \theta_0, \qquad \rho(x_0) = \rho_0,$$

exists on any interval containing x_0 in which $p(x)$ is bounded away from zero.

Solution

- Using (2) in (1), we get

  $$-(\rho\cos\theta)' + q\rho\sin\theta = \lambda r\rho\sin\theta,$$

  or

  $$-\rho'\cos\theta + \rho(\sin\theta)\theta' = (\lambda r - q)\rho\sin\theta. \tag{4}$$

  The transformation (2) implies that

  $$p(\rho\sin\theta)' = \rho\cos\theta$$

  or

  $$p\rho'\sin\theta + p\rho(\cos\theta)\theta' = \rho\cos\theta. \tag{5}$$

  Solving (4)–(5) for θ' and ρ' (multiply (4) by $p\sin\theta$ and (5) by $\cos\theta$ and add to get θ'; multiply (4) by $p\cos\theta$ and (5) by $\sin\theta$ and subtract to get ρ') we obtain (3).

- Note that the equation for θ does not involve ρ, so we can solve it first, then use the result in the equation for ρ. The θ-equation is a nonlinear first-order ODE, but the right hand side is a bounded function of θ (no blow-up is possible!) so its solutions exist for all x provided that $1/p$ is continuous. Given a solution for θ, the equation for ρ is a linear first order ODE whose solutions also exist for all x.

Remark. This transformation to polar coordinates in the (u, pu') phase-plane is called the Prüfer transformation. It can be used to prove the basic properties of Sturm-Liouville eigenvalue problems, such as the existence of infinitely many eigenvalues, and the oscillation theorems. For example, to study the Sturm-Liouville equation with Dirichlet BCs $u(0) = u(1) = 0$, we impose the initial condition $\theta(0) = 0$ and look for values of λ such that $\theta(1) = n\pi$ for some integer n.