Sample Final Questions

Math 207B, Winter 2012

1. Find an explicit expression for the Green's function for the problem

$$\begin{aligned} &-u'' + u = f(x), \qquad 0 < x < 1 \\ &u'(0) = 0, \qquad u'(1) = 0. \end{aligned}$$

Write down the Green's function representation of the solution for $u(x)$.

2. Use separation of variables and Fourier series to solve the following IBVP for the Schrödinger equation for the complex-valued function $\psi(x,t)$

$$\begin{aligned} &i\psi_t = -\psi_{xx}, \qquad 0 < x < 1 \\ &\psi(0,t) = 0, \qquad \psi(1,t) = 0, \\ &\psi(x,0) = f(x) \end{aligned}$$

where $f \in L^2(0,1)$ is given initial data. Show from your solution that

$$\int_0^1 |\psi(x,t)|^2 \, dx = \int_0^1 |f(x)|^2 \, dx \qquad \text{for all } t \in \mathbb{R}.$$

3. After non-dimensionalization, the displacement $u(x)$ of a non-uniform string, with density $\rho(x)$, fixed at each end and vibrating with frequency ω satisfies the EVP

$$\begin{aligned} &-u'' = \lambda\rho(x)u, \qquad 0 < x < 1, \\ &u(0) = 0, \qquad u(1) = 0 \end{aligned}$$

where $\lambda = \omega^2$. The fundamental frequency of the string is $\omega_1 = \sqrt{\lambda_1}$, where $\lambda = \lambda_1$ is the smallest eigenvalue. If $m \leq \rho(x) \leq M$ where m, M are positive constants, show that

$$\frac{\pi}{\sqrt{M}} \leq \omega_1 \leq \frac{\pi}{\sqrt{m}}.$$

Does this result make sense physically?

4. Consider the Volterra integral operator $K : L^2(0,1) \to L^2(0,1)$ defined by

$$Ku(x) = \int_0^x u(y)\,dy, \qquad 0 < x < 1$$

Show that the integral equation $Ku = \lambda u$ has no nonzero solutions for any $\lambda \in \mathbb{C}$, meaning that K has no eigenvalues. Why doesn't this contradict the spectral theorem for compact (or Hilbert-Schmidt) self-adjoint operators?

5. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded region, and define an operator L by

$$Lu = -\nabla \cdot (p\nabla u) + qu$$

where p, q are smooth functions on $\bar{\Omega}$. Show that

$$\int_\Omega uLv\,dx = \int_\Omega vLu\,dx$$

for all functions $u, v : \Omega \to \mathbb{R}$ that vanish on the boundary $\partial\Omega$, meaning that L with Dirichlet BCs is formally self-adjoint.

6. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded region, Consider the Neumann BVP

$$\begin{aligned} -\Delta u &= f(x) \qquad x \in \Omega, \\ \frac{\partial u}{\partial n} &= g(x) \qquad x \in \partial\Omega. \end{aligned}$$

(a) Show that a solution can only exist if

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,dS = 0$$

Give a physical interpretation of this result in terms of heat flow.

(b) If a solution exists, show that it is unique up to an arbitrary additive constant.