PROBLEM SET 1
Math 207B, Winter 2012
Due: Fri., Jan. 20

1. If $\lambda \le 0$, show that there are no non-zero solutions for $u(x)$ of the Sturm-Liouville problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u(0) = 0, &\qquad u(1) = 0. \end{aligned}$$

2. Find all eigenvalues λ_n and eigenfunctions $u_n(x)$ of the Sturm-Liouville problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u'(0) = 0, &\qquad u'(1) = 0 \end{aligned}$$

with Neumann boundary conditions. Show that

$$\int_0^1 u_n(x) u_m(x)\, dx = 0$$

if $n \neq m$.

3. Consider a wave equation

$$u_{tt} - \left(c_0^2 u_x\right)_x + q_0 u = 0$$

for $u(x,t)$ where the wave speed $c_0(x)$ and the coefficient $q_0(x)$ depend on x. Look for (possibly complex-valued) separable solutions of the form

$$u(x,t) = X(x)T(t).$$

Show that, up to linear superpositions, we can take

$$T(t) = e^{-i\omega t}$$

for some constant $\omega \in \mathbb{C}$. Find the corresponding ODE satisfied by X in that case. (Note that if a function of x is equal to a function of t, then the functions must be constant.)

4. Define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right)$$

and let

$$\phi_n(x) = e^{-x^2/2} H_n(x)$$

where $n = 0, 1, 2, 3, \ldots$. Show that ϕ_n is a solution of the Sturm-Liouville problem

$$\begin{aligned} &-\phi_n'' + x^2 \phi_n = \lambda_n \phi_n \qquad -\infty < x < \infty, \\ &\phi_n(x) \to 0 \quad \text{as } |x| \to \infty \end{aligned}$$

with eigenvalue

$$\lambda_n = 2n + 1.$$

Hint: Let L be the linear operator

$$L = -\frac{d^2}{dx^2} + x^2,$$

meaning that L acts on functions ϕ by

$$L\phi = -\phi'' + x^2 \phi.$$

Define operators A, A^* by

$$A = \frac{d}{dx} + x, \qquad A^* = -\frac{d}{dx} + x.$$

Show that

$$A\phi_n = 2n\phi_{n-1}, \quad A^*\phi_n = \phi_{n+1}$$

and

$$L = AA^* - 1.$$