SOLUTIONS: PROBLEM SET 1
Math 207B, Winter 2012

1. If $\lambda \leq 0$, show that there are no non-zero solutions for $u(x)$ of the Sturm-Liouville problem

$$
\begin{aligned}
-u'' &= \lambda u \qquad 0 < x < 1, \\
u(0) &= 0, \qquad u(1) = 0.
\end{aligned}
$$

Solution

- If $\lambda < 0$, with $\lambda = -k^2$ for $k > 0$ say, then the general solution of the ODE is
$$u(x) = c_1 \cosh kx + c_2 \sinh kx.$$
The BC $u(0) = 0$ implies that $c_1 = 0$, and then the BC $u(1) = 0$ implies that $c_2 = 0$, since $\sinh k \neq 0$. Therefore $u = 0$ and λ is not an eigenvalue.

- If $\lambda = 0$, then the general solution of the ODE is
$$u(x) = c_1 + c_2 x$$
and the BCs imply that $c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.

2. Find all eigenvalues λ_n and eigenfunctions $u_n(x)$ of the Sturm-Liouville problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u'(0) &= 0, \qquad u'(1) = 0 \end{aligned}$$

with Neumann boundary conditions. Show that

$$\int_0^1 u_n(x) u_m(x)\, dx = 0$$

if $n \neq m$.

Solution

- The problem is self-adjoint so all eigenvalues must be real.
- One can check as in Problem 1 that if $\lambda < 0$, then the only solution is $u = 0$ so λ is not an eigenvalue.
- If $\lambda = 0$, then $u =$ constant is a solution, so $\lambda_0 = 0$ is an eigenvalue with eigenfunction $u_0 = 1$.
- If $\lambda = k^2 > 0$, then $u(x) = \cos kx$ is a solution of the ODE with $u'(0) = 0$. This function satisfies the BC $u'(1) = 0$ if $\sin k = 0$ or $k = n\pi$ for $n \in \mathbb{N}$.
- The eigenvalues are

  $$\lambda_n = n^2\pi^2 \qquad \text{for } n = 0, 1, 2, \ldots$$

  with eigenfunctions

  $$u_n(x) = \cos(n\pi x) \qquad \text{for } n = 0, 1, 2, \ldots.$$

3. Consider a wave equation

$$u_{tt} - \left(c_0^2 u_x\right)_x + q_0 u = 0$$

for $u(x,t)$ where the wave speed $c_0(x)$ and the coefficient $q_0(x)$ depend on x. Look for (possibly complex-valued) separable solutions of the form

$$u(x,t) = X(x)T(t).$$

Show that, up to linear superpositions, we can take

$$T(t) = e^{-i\omega t}$$

for some constant $\omega \in \mathbb{C}$. Find the corresponding ODE satisfied by X in that case. (Note that if a function of x is equal to a function of t, then the functions must be constant.)

Solution

- Using $u(x,t) = X(x)T(t)$ in the wave equation, we get

  $$XT'' - \left(c_0^2 X'\right)' T + q_0 XT = 0.$$

  Dividing this equation by XT and rearranging the result, we get that

  $$\frac{T''}{T} = \frac{\left(c_0^2 X'\right)'}{X} - q_0.$$

- The left-hand side is function of t only and the right-hand side is a function of x only, so each function must be constant, equal to $\lambda = -\omega^2$, say, for some $\omega \in \mathbb{C}$.

- It follows that

  $$T'' + \omega^2 T = 0$$

  so (if $\omega \neq 0$)

  $$T(t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t}.$$

- The corresponding Sturm-Liouville ODE for $X(x)$ is

  $$-\left(c_0^2 X'\right)' + q_0 X = \omega^2 X.$$

4. Define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right)$$

and let

$$\phi_n(x) = e^{-x^2/2} H_n(x)$$

where $n = 0, 1, 2, 3, \ldots$. Show that ϕ_n is a solution of the Sturm-Liouville problem

$$\begin{aligned} &-\phi_n'' + x^2 \phi_n = \lambda_n \phi_n \qquad -\infty < x < \infty, \\ &\phi_n(x) \to 0 \quad \text{as } |x| \to \infty \end{aligned}$$

with eigenvalue

$$\lambda_n = 2n + 1.$$

Hint: Let L be the linear operator

$$L = -\frac{d^2}{dx^2} + x^2,$$

meaning that L acts on functions ϕ by

$$L\phi = -\phi'' + x^2 \phi.$$

Define operators A, A^* by

$$A = \frac{d}{dx} + x, \qquad A^* = -\frac{d}{dx} + x.$$

Show that

$$A\phi_n = 2n\phi_{n-1}, \quad A^*\phi_n = \phi_{n+1} \tag{1}$$

and

$$L = AA^* - 1.$$

Solution

- Note that

  $$(xu)' = xu' + u,$$

  which means that

  $$\frac{d}{dx} x = x \frac{d}{dx} + 1,$$

or

$$\left[\frac{d}{dx}, x\right] = \frac{d}{dx}x - x\frac{d}{dx} = 1.$$

Hence

$$\begin{aligned}
AA^* &= \left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right) \\
&= -\frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{d}{dx}x + x^2 \\
&= -\frac{d^2}{dx^2} + x^2 + 1 \\
&= L + 1.
\end{aligned}$$

- Assuming (1), we get that

$$\begin{aligned}
L\phi_n &= (AA^* - 1)\,\phi_n \\
&= AA^*\phi_n - \phi_n \\
&= A\phi_{n+1} - \phi_n \\
&= 2(n+1)\phi_n - \phi_n \\
&= (2n+1)\phi_n,
\end{aligned}$$

which shows that ϕ_n is an eigenfunction of L with eigenvalue $2n+1$.

- We have

$$\phi_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

Hence

$$\begin{aligned}
A^*\phi_n &= (-1)^n\left(-\frac{d}{dx} + x\right)\left[e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right] \\
&= (-1)^{n+1}\frac{d}{dx}\left[e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right] \\
&\qquad + (-1)^n x e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) \\
&= (-1)^{n+1}e^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) \\
&= \phi_{n+1}.
\end{aligned}$$

- We have

$$A\phi_n = (-1)^n \left(\frac{d}{dx} + x\right) \left[e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2}\right)\right]$$
$$= (-1)^n e^{x^2/2} \left[\frac{d^{n+1}}{dx^{n+1}} \left(e^{-x^2}\right) + 2x \frac{d^n}{dx^n} \left(e^{-x^2}\right)\right].$$

Since derivatives of x of the order greater than or equal to two vanish, the Leibnitz formula

$$\frac{d^n}{dx^n}(fg) = f\frac{d^n g}{dx^n} + n\frac{df}{dx}\frac{d^{n-1}g}{dx^{n-1}} + \frac{1}{2}n(n-1)\frac{d^2 f}{dx^2}\frac{d^{n-2}g}{dx^{n-2}} + \cdots + \frac{d^n f}{dx^n}g$$

implies that

$$\frac{d^{n+1}}{dx^{n+1}} \left(e^{-x^2}\right) = -2\frac{d^n}{dx^n} \left(xe^{-x^2}\right)$$
$$= -2\left[x\frac{d^n}{dx^n} \left(e^{-x^2}\right) + n\frac{d^{n-1}}{dx^{n-1}} \left(e^{-x^2}\right)\right].$$

It follows that

$$A\phi_n = 2n(-1)^{n-1} e^{x^2/2} \frac{d^{n-1}}{dx^{n-1}} \left(e^{-x^2}\right)$$
$$= 2n\phi_{n-1}$$

which completes the proof of (1).