PROBLEM SET 2
Math 207B, Winter 2012
Due: Fri., Jan. 27

1. Suppose that L is the second-order differential operator

$$L = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)$$

Find the adjoint operator

$$L^* = q_2(x)\frac{d^2}{dx^2} + q_1(x)\frac{d}{dx} + q_0(x)$$

such that

$$\int_a^b (uL^*v - vLu)\ dx = 0$$

for all functions $u, v : [a, b] \to \mathbb{R}$ that satisfy

$$u(a) = u'(a) = 0,\ u(b) = u'(b) = 0, \quad v(a) = v'(a) = 0,\ v(b) = v'(b) = 0.$$

Show that the Sturm-Liouville operator

$$L = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x)$$

is the most general formally self-adjoint operator (meaning that $L = L^*$).

2. Consider the Sturm-Liouville operator

$$L = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x)$$

acting on functions $u(x)$ defined on the interval $a \le x \le b$. Find adjoint boundary conditions in the following cases. (a) Neumann $u'(a) = u'(b) = 0$; (b) Periodic $u(a) = u(b)$, $u'(a) = u'(b)$; (c) Initial $u(a) = u'(a) = 0$. Which boundary conditions are self-adjoint?

3. If u_1, u_2 are two solutions of the Sturm-Liouville equation

$$-(pu')' + qu = \lambda u$$

show that

$$p\left(u_1 u_2' - u_2 u_1'\right) = \text{constant}.$$

Deduce that if $p(x) > 0$ on $[a, b]$ and u_1, u_2 satisfy the separated BCs

$$(\cos\alpha)u(a) + (\sin\alpha)u'(a) = 0, \qquad (\cos\beta)u(b) + (\sin\beta)u'(b) = 0$$

for some constants α, β then u_1, u_2 are linearly dependent. (It follows that all eigenvalues are simple.)

4. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 2\pi, \\ u(0) &= u(2\pi), \qquad u'(0) = u'(2\pi) \end{aligned}$$

with periodic boundary conditions. Verify explicitly that eigenfunctions with different eigenvalues are orthogonal. Are the eigenvalues simple? Is this answer consistent with the result of Problem 3?