Solutions: Problem set 2
Math 207B, Winter 2012

1. Suppose that L is the second-order differential operator

$$L = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)$$

Find the adjoint operator

$$L^* = q_2(x)\frac{d^2}{dx^2} + q_1(x)\frac{d}{dx} + q_0(x)$$

such that

$$\int_a^b (uL^*v - vLu)\ dx = 0$$

for all functions $u, v : [a, b] \to \mathbb{R}$ that satisfy

$$u(a) = u'(a) = 0,\ u(b) = u'(b) = 0, \quad v(a) = v'(a) = 0,\ v(b) = v'(b) = 0.$$

Show that the Sturm-Liouville operator

$$L = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x)$$

is the most general formally self-adjoint operator (meaning that $L = L^*$).

Solution

- Integration by parts gives (all the boundary terms vanish)

$$\begin{aligned}\int_a^b vLu\, dx &= \int_a^b v\left(p_2u'' + p_1u' + p_0u\right)\, dx\\ &= \int_a^b u\left[(p_2v)'' - (p_1v)' + p_0v\right]\, dx\\ &= \int_a^b uL^*v\, dx\end{aligned}$$

  where

$$\begin{aligned}L^*v &= (p_2v)'' - (p_1v)' + p_0v\\ &= p_2v'' + \left(2p_2' - p_1\right)v' + \left(p_2'' - p_1' + p_0\right)v.\end{aligned}$$

  Therefore

$$q_2 = p_2, \qquad q_1 = 2p_2' - p_1, \qquad q_0 = p_2'' - p_1' + p_0.$$

- In order for L to be self adjoint, we need $q_i = p_i$, which is the case if $p_1 = p_2'$. Writing $p_2 = -p$, $p_1 = -p'$, and $p_0 = q$, we see that the most general (second-order, real, scalar, ordinary, linear) formally self-adjoint differential operator has the form

  $$Lu = -pu'' - p'u' + qu = -(pu')' + qu,$$

  which is the Sturm-Liouville operator.

2. Consider the Sturm-Liouville operator

$$L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x)$$

acting on functions $u(x)$ defined on the interval $a \le x \le b$. Find adjoint boundary conditions in the following cases. (a) Neumann $u'(a) = u'(b) = 0$; (b) Periodic $u(a) = u(b)$, $u'(a) = u'(b)$; (c) Initial $u(a) = u'(a) = 0$. Which boundary conditions are self-adjoint?

Solution

- Integrating by parts, we have

  $$\int_a^b (uLv - vLu)\, dx = \left[p\,(vu' - uv')\right]_a^b .$$

  The adjoint boundary conditions $B^*(v) = 0$ to $B(u) = 0$ are defined to be most general conditions on v which ensure that

  $$\left[p\,(vu' - uv')\right]_a^b = 0$$

  for all functions u such that $B(u) = 0$.

- (a) If $u'(a) = u'(b) = 0$, then

  $$\left[p\,(vu' - uv')\right]_a^b = -p(b)u(b)v'(b) + p(a)u(a)v'(a).$$

  This vanishes for all values of $u(a)$, $u(b)$ — which can be anything — if and only if

  $$v'(a) = v'(b) = 0.$$

  (We assume here that $p(a)$, $p(b)$ are non-zero.) Neumann BCs are self-adjoint.

- (b) If $u(a) = u(b)$, $u'(a) = u'(b)$, then

  $$\left[p\,(vu' - uv')\right]_a^b = p(a)u'(a)\left[v(b) - v(a)\right] - p(a)u(a)\left[v'(b) - v'(a)\right].$$

  (We assume here that the coefficient p is a non-vanishing, continuous periodic function, so $p(a) = p(b) \neq 0$.) Thus, the adjoint BCs are

  $$v(a) = v(b), \qquad v'(a) = v'(b)$$

  and periodic BCs are self-adjoint.

- (c) If $u(a) = u'(a) = 0$, then

  $$[p\,(vu' - uv')]_a^b = p(b)\,[v(b)u'(b) - u(b)v'(b)]\,.$$

  Assuming $p(b) \neq 0$, we see that this vanishes for all values of $u(b)$, $u'(b)$ if and only if

  $$v(b) = v'(b) = 0.$$

  Thus the adjoint of initial value conditions (ICs) at $x = a$ is final value conditions at $x = b$, and ICs are not self-adjoint.

3. If u_1, u_2 are two solutions of the Sturm-Liouville equation

$$-(pu')' + qu = \lambda u$$

show that

$$p\left(u_1 u_2' - u_2 u_1'\right) = \text{constant}. \tag{1}$$

Deduce that if $p(x) > 0$ on $[a, b]$ and u_1, u_2 satisfy the separated BCs

$$(\cos\alpha)u(a) + (\sin\alpha)u'(a) = 0, \qquad (\cos\beta)u(b) + (\sin\beta)u'(b) = 0 \tag{2}$$

for some constants α, β then u_1, u_2 are linearly dependent. (It follows that all eigenvalues are simple.)

Solution

- Using the product rule and the assumption that u_1, u_2 are solutions of the ODE, we have

$$\begin{aligned}\frac{d}{dx}\left[p\left(u_1 u_2' - u_2 u_1'\right)\right] &= \frac{d}{dx}\left[u_1 \cdot pu_2' - u_2 \cdot pu_1'\right] \\ &= u_1(pu_2')' + u_1' \cdot pu_2' - u_2 \cdot (pu_1')' - u_2' \cdot pu_1' \\ &= u_1(pu_2')' - u_2 \cdot (pu_1')' \\ &= u_1(q-\lambda)u_2 - u_2(q-\lambda)u_1 \\ &= 0,\end{aligned}$$

  which implies (1).

- Let

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1'$$

  denote the Wronskian of u_1, u_2. Note that

$$\begin{aligned}u_1 u_2' - u_2 u_1' = &\left(\cos\alpha u_1 + \sin\alpha u_1'\right)\left(-\sin\alpha u_2 + \cos\alpha u_2'\right) \\ &- \left(\cos\alpha u_2 + \sin\alpha u_2'\right)\left(-\sin\alpha u_1 + \cos\alpha u_1'\right).\end{aligned}$$

  Thus, if u_1, u_2 satisfy (2), then their Wronskian vanishes at $x = a, b$ and therefore by the previous result it is identically zero on $[a, b]$, which implies that u_1, u_2 are linearly dependent.

4. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$-u'' = \lambda u \qquad 0 < x < 2\pi,$$
$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi)$$

with periodic boundary conditions. Verify explicitly that eigenfunctions with different eigenvalues are orthogonal. Are the eigenvalues simple? Is this answer consistent with the result of Problem 3?

Solution

- The eigenvalues are
$$\lambda_n = n^2, \qquad n = 0, 1, 2, \dots.$$
- For $\lambda = 1$, there is a one-dimensional space of eigenfunctions spanned by the constant function
$$u_0(x) = 1.$$
- For $\lambda = n^2$ with $n \geq 1$, there is a two-dimensional space of eigenfunctions spanned by the complex exponentials
$$u_{-n}(x) = e^{-inx}, \qquad u_n(x) = e^{inx}.$$
(Alternatively, we could use the real-valued trigonometric functions $\cos nx$, $\sin nx$ as a basis of the eigenspace.)
- For $m \neq n$, we have
$$\begin{aligned}\left(e^{imx}, e^{inx}\right) &= \int_0^{2\pi} e^{imx} e^{-inx}\, dx \\ &= \int_0^{2\pi} e^{i(m-n)x}\, dx \\ &= \left[\frac{e^{i(m-n)x}}{i(m-n)}\right]_0^{2\pi} \\ &= 0\end{aligned}$$
by periodicity.
- All eigenvalues but the lowest eigenvalue $\lambda = 0$ are not simple (they have multiplicity two). Periodic boundary conditions are self-adjoint but not separable, so the fact that the eigenvalues are not simple does not contradict the result of Problem 3.