Problem Set 3
Math 207B, Winter 2012
Due: Fri., Feb. 3

1. Suppose that $u(x)$ is a solution of the Sturm-Liouville problem with nonhomogeneous ODE and BCs

$$\begin{aligned} &-(pu')' + qu = f(x) \qquad a < x < b, \\ &u(a) = A, \qquad u(b) = B. \end{aligned}$$

Write

$$u(x) = A\left(\frac{b-x}{b-a}\right) + B\left(\frac{x-a}{b-a}\right) + v(x)$$

and show that v satisfies a Sturm-Liouville problem of the form

$$\begin{aligned} &-(pv')' + qv = g(x) \qquad a < x < b, \\ &v(a) = 0, \qquad v(b) = 0 \end{aligned}$$

with homogeneous BCs.

2. Consider the nonhomogeneous Sturm-Liouville problem

$$\begin{aligned} &-(pu')' + qu = \lambda u + f(x) \qquad a < x < b, \\ &u(a) = 0, \qquad u(b) = 0. \end{aligned}$$

If λ is an eigenvalue with eigenfunction ϕ, show that the problem only has a solution if f satisfies

$$\int_a^b f\overline{\phi}\, dx = 0.$$

Under what conditions on f is the BVP

$$\begin{aligned} &-u'' = f(x) \qquad 0 < x < 1, \\ &u'(0) = 0, \qquad u'(1) = 0 \end{aligned}$$

solvable? How about the BVP

$$\begin{aligned} &-u'' = f(x) \qquad 0 < x < 1, \\ &u'(0) = 0, \qquad u'(1) = 1. \end{aligned}$$

3. Consider the weighted Sturm-Liouville eigenvalue problem

$$\begin{aligned} &-(pu')' + qu = \lambda r u \qquad a < x < b, \\ &u(a) = 0, \qquad u(b) = 0 \end{aligned}$$

where $p(x)$, $q(x)$, $r(x)$ are given real-valued coefficient functions and $r > 0$. Let $L^2_r(a, b)$ denote the space of functions $f : [a, b] \to \mathbb{C}$ such that

$$\int_a^b r|f|^2\, dx < \infty$$

with weighted inner product

$$(f, g)_r = \int_a^b r f \overline{g}\, dx.$$

(a) If $\phi(x)$ is an eigenfunction with eigenvalue $\lambda \in \mathbb{C}$, show that $\lambda \in \mathbb{R}$ is real.

(b) If $\phi(x)$, $\psi(x)$ are eigenfunctions with distinct eigenvalues λ, μ show that they are orthogonal with respect to the weighted inner-product, meaning that

$$\int_a^b r\phi\overline{\psi}\, dx = 0.$$

(c) Suppose that the eigenvalue problem has a complete set of eigenfunctions $\{\phi_n : n = 1, 2, 3, \dots\}$. If $f \in L^2_r(a, b)$, give an expression for the coefficients c_n in the eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

4. Use separation of variables to solve the following IBVP for $u(x, t)$ for the wave equation:

$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < 1, \\ &u_x(0, t) = 0, && u(1, t) = 0, \\ &u(x, 0) = f(x), && u_t(x, 0) = g(x). \end{aligned}$$