Solutions: Problem set 3

Math 207B, Winter 2012

1. Suppose that $u(x)$ is a solution of the Sturm-Liouville problem with nonhomogeneous ODE and BCs

$$
\begin{aligned}
&-(pu')' + qu = f(x) \qquad a < x < b,\\
&u(a) = A, \qquad u(b) = B.
\end{aligned}
$$

Write

$$u(x) = A\left(\frac{b-x}{b-a}\right) + B\left(\frac{x-a}{b-a}\right) + v(x)$$

and show that v satisfies a Sturm-Liouville problem of the form

$$
\begin{aligned}
&-(pv')' + qv = g(x) \qquad a < x < b,\\
&v(a) = 0, \qquad v(b) = 0
\end{aligned}
$$

with homogeneous BCs.

Solution

- Write the ODE as

$$Lu = f, \qquad L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q.$$

  Let $u(x) = u_0(x) + v(x)$. Then since L is linear $Lu = Lu_0 + Lv$, and therefore

$$Lv = g, \qquad g = f - Lu_0.$$

  Moreover if $u_0(a) = A$, $u_0(b) = B$, then $v(a) = v(b) = 0$.

- Apply this result with the given u_0.

Remark. For a linear problem with nonhomogeneous boundary conditions, we can subtract off any function that satisfies the boundary conditions and transfer the nonhomogeneity to the ODE.

2. Consider the nonhomogeneous Sturm-Liouville problem

$$-(pu')' + qu = \lambda u + f(x) \qquad a < x < b,$$
$$u(a) = 0, \qquad u(b) = 0.$$

If λ is an eigenvalue with eigenfunction ϕ, show that the problem only has a solution if f satisfies

$$\int_a^b f\overline{\phi}\, dx = 0.$$

Under what conditions on f is the BVP

$$-u'' = f(x) \qquad 0 < x < 1,$$
$$u'(0) = 0, \qquad u'(1) = 0$$

solvable? How about the BVP

$$-u'' = f(x) \qquad 0 < x < 1,$$
$$u'(0) = 0, \qquad u'(1) = 1.$$

Solution

- Write the ODE as

$$(L - \lambda I)u = f$$

and assume that there is a solution u. Taking the inner product of this equation with ϕ, we get

$$((L - \lambda I)u, \phi) = (f, \phi).$$

Since L is self-adjoint, any eigenvalue λ is real and $(L - \lambda I)^* = L - \lambda I$ is self-adjoint. (If $\lambda \in \mathbb{C}$ is complex, then $(L - \lambda I)^* = L - \bar{\lambda} I$.) Hence, since u, ϕ satisfy self-adjoint BCs, we have

$$(f, \phi) = (u, (L - \lambda I)\phi) = 0.$$

- If $L = -d^2/dx^2$, with Neumann BCs, then $\lambda = 0$ is an eigenvalue with eigenfunction $\phi = 1$. If follows that the equation $Lu = f$, with BCs $u'(0) = u'(1) = 0$ is only solvable if $(f, 1) = 0$, or

$$\int_0^1 f(x)\, dx = 0.$$

- We can verify this condition directly: if $-u'' = f(x)$ and $u'(0) = u'(1) = 0$, then

$$\int_0^1 f(x)\,dx = -\int_0^1 u''\,dx = [u']_0^1 = 0.$$

- If $-u'' = f(x)$ and $u'(0) = 0$, $u'(1) = 1$, let

$$u(x) = \frac{1}{2}x^2 + v(x).$$

  Then

$$-v'' = -u'' + 1 = f(x) + 1, \qquad v'(0) = v'(1) = 0.$$

  Hence the equation is only solvable if

$$\int_0^1 [f(x) + 1]\;dx = 0$$

  or

$$\int_0^1 f(x)\,dx = -1.$$

  Alternatively, as a direct verification,

$$\int_0^1 f(x)\,dx = -\int_0^1 u''\,dx = [u']_0^1 = -1.$$

Remark. In general, a necessary condition for the solvability of a singular linear equation $Lu = f$ is that f is orthogonal to the right null space of the adjoint L^*.

3. Consider the weighted Sturm-Liouville eigenvalue problem

$$
\begin{aligned}
&-(pu')' + qu = \lambda r u \qquad a < x < b, \\
&u(a) = 0, \qquad u(b) = 0
\end{aligned}
$$

where $p(x)$, $q(x)$, $r(x)$ are given real-valued coefficient functions and $r > 0$. Let $L_r^2(a,b)$ denote the space of functions $f : [a,b] \to \mathbb{C}$ such that

$$\int_a^b r|f|^2\,dx < \infty$$

with weighted inner product

$$(f,g)_r = \int_a^b r f \overline{g}\,dx.$$

(a) If $\phi(x)$ is an eigenfunction with eigenvalue $\lambda \in \mathbb{C}$, show that $\lambda \in \mathbb{R}$ is real.

(b) If $\phi(x)$, $\psi(x)$ are eigenfunctions with distinct eigenvalues λ, μ show that they are orthogonal with respect to the weighted inner-product, meaning that

$$\int_a^b r\phi\overline{\psi}\,dx = 0.$$

(c) Suppose that the eigenvalue problem has a complete set of eigenfunctions $\{\phi_n : n = 1, 2, 3, \dots\}$. If $f \in L_r^2(a,b)$, give an expression for the coefficients c_n in the eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Solution

- Suppose that

$$L\phi = \lambda r \phi, \quad L\psi = \lambda r \psi, \qquad \phi(a) = \phi(b) = 0,\ \psi(a) = \psi(b) = 0$$

where $\lambda, \mu \in \mathbb{C}$ and

$$L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q.$$

Using the self-adjointness of L on $L^2(a,b)$ with Dirichlet BCs, we have

$$\begin{aligned}\lambda(\phi,\psi)_r &= \int_a^b \lambda r\phi\overline{\psi}\,dx = \int_a^b (L\phi)\overline{\psi}\,dx \\ &= \int_a^b \phi\overline{(L\psi)}\,dx = \int_a^b \phi\overline{(\mu r\psi)}\,dx = \overline{\mu}(\phi,\psi)_r.\end{aligned}$$

- (a) If $\phi = \psi$ and $\lambda = \mu$ then, since $(\phi,\phi)_r > 0$, we conclude that $\lambda = \overline{\lambda}$ so $\lambda \in \mathbb{R}$.
- (b) If $\lambda \neq \mu$ then
$$\lambda(\phi,\psi)_r = \mu(\phi,\psi)_r$$
so $(\phi,\psi)_r = 0$.
- (c) Taking the weighted inner product of the series for f, and using the the orthogonality of the ϕ_n, we find that
$$\int_a^b rf\overline{\phi_n}\,dx = c_n \int_a^b r|\phi_n|^2\,dx,$$
which gives
$$c_n = \frac{\int_a^b rf\overline{\phi_n}\,dx}{\int_a^b r|\phi_n|^2\,dx}.$$

4. Use separation of variables to solve the following IBVP for $u(x,t)$ for the wave equation:

$$
\begin{aligned}
&u_{tt} = u_{xx} && 0 < x < 1, \\
&u_x(0,t) = 0, && u(1,t) = 0, \\
&u(x,0) = f(x), && u_t(x,0) = g(x).
\end{aligned}
$$

Solution

- Look for separable solutions of the form

$$u(x,t) = X(x)T(t).$$

Then

$$X\ddot{T} = X''T$$

so

$$\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

where λ is a separation constant.

- Imposing the BCs on X, we get the Sturm-Liouville problem

$$-X'' = \lambda X, \qquad X'(0) = 0, \quad X(1) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = \pi^2 \left(n + \frac{1}{2}\right)^2, \qquad X_n(x) = \cos\left[\pi\left(n + \frac{1}{2}\right)x\right]$$

for $n = 0, 1, 2, \ldots$.

- The corresponding functions T_n satisfy

$$\ddot{T}_n + \pi^2\left(n + \frac{1}{2}\right)^2 T_n = 0$$

whose solution is

$$T_n(t) = a_n \cos\left[\pi\left(n + \frac{1}{2}\right)t\right] + b_n \sin\left[\pi\left(n + \frac{1}{2}\right)t\right].$$

- Superposing these solutions, we get as a solution of the PDE

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left[\pi\left(n+\frac{1}{2}\right)t\right]\cos\left[\pi\left(n+\frac{1}{2}\right)x\right]$$
$$+\sum_{n=0}^{\infty} b_n \sin\left[\pi\left(n+\frac{1}{2}\right)t\right]\cos\left[\pi\left(n+\frac{1}{2}\right)x\right].$$

- By completeness of the eigenfunctions, the initial conditions are satisfied if

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left[\pi\left(n+\frac{1}{2}\right)x\right],$$
$$g(x) = \sum_{n=0}^{\infty} b_n \pi\left(n+\frac{1}{2}\right)\cos\left[\pi\left(n+\frac{1}{2}\right)x\right].$$

- Using the orthogonality relations

$$\int_0^1 \cos\left[\pi\left(m+\frac{1}{2}\right)x\right]\cos\left[\pi\left(n+\frac{1}{2}\right)x\right]\,dx$$
$$= \frac{1}{2}\int_0^1 \left\{\cos\left[\pi\left(m+n+1\right)x\right]+\cos\left[\pi\left(m-n\right)x\right]\right\}\,dx$$
$$= \begin{cases} 1/2 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

we get

$$a_n = 2\int_0^1 f(x)\cos\left[\pi\left(n+\frac{1}{2}\right)x\right]\,dx,$$
$$b_n = \frac{4}{\pi\left(2n+1\right)}\int_0^1 g(x)\cos\left[\pi\left(n+\frac{1}{2}\right)x\right]\,dx.$$