Problem Set 4
Math 207B, Winter 2012
Due: Fri., Feb. 10

1. (a) Consider the 2π-periodic function $f(x;\epsilon)$ defined for $\epsilon > 0$ by

$$f(x;\epsilon) = \begin{cases} 1/\epsilon & \text{if } 0 < x < \epsilon, \\ 0 & \text{if } \epsilon < x < 2\pi. \end{cases}, \qquad f(x+2\pi;\epsilon) = f(x;\epsilon).$$

Sketch the graph of $f(x,\epsilon)$ on the real line. Compute its Fourier series

$$f(x;\epsilon) = \sum_{n=-\infty}^{\infty} f_n(\epsilon) e^{inx}.$$

(b) Define the 2π-periodic δ-function δ_p, or 'δ-comb', by

$$\delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n)$$

where $\delta(x)$ is δ-function on the real line supported at $x = 0$. Draw a picture of δ_p. Show that a formal computation of the Fourier coefficients of δ_p gives

$$\delta_p(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx}.$$

(c) Show that you recover the Fourier series in (b) by taking the limit as $\epsilon \to 0^+$ of the Fourier series in (a).

2. Consider the 2π-periodic square wave S defined by

$$S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ 0 & \text{if } -\pi < x < 0. \end{cases}, \qquad S(x+2\pi) = S(x).$$

(a) Sketch the graph of S on the real line and explain why the (distributional) derivative of S is given by

$$S'(x) = \delta_p(x) - \delta_p(x - \pi)$$

where δ_p is the periodic δ-function from Problem 1.

(b) Compute the Fourier series of S

$$S(x) = \sum_{n=-\infty}^{\infty} S_n e^{inx}.$$

(c) Show that the formal term-by-term derivative of this series agrees with the Fourier series of $\delta_p(x) - \delta_p(x-\pi)$ from Problem 1.

3. Consider the non-homogeneous Sturm-Liouville problem

$$\begin{aligned} &-u'' = \lambda u + f(x) \qquad 0 < x < 1, \\ &u'(0) = 0, \qquad u(1) = 0 \end{aligned} \tag{1}$$

where $\lambda = k^2$ with $k > 0$ is a strictly positive constant.

(a) Give the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the homogeneous problem, $n = 1, 2, 3, \ldots$, which satisfy

$$\begin{aligned} &-\phi_n'' = \lambda_n \phi_n \qquad 0 < x < 1, \\ &\phi_n'(0) = 0, \qquad \phi_n(1) = 0. \end{aligned}$$

(b) Write out the eigenfunction expansion for the Green's function $G(x, \xi; \lambda)$.

(b) Find an explicit expression for the Green's function $G(x, \xi; \lambda)$ in terms of appropriate solutions of the homogeneous equation. Show that your solution has poles at the eigenvalues $\lambda = \lambda_n$ and that its residues give the eigenfunctions.

4. Consider the non-homogeneous Sturm-Liouville problem on the real line

$$\begin{aligned} &-u'' + u = f(x) \qquad -\infty < x < \infty, \\ &u(x) \to 0 \qquad \text{as } x \to \pm\infty \end{aligned} \tag{2}$$

where $f(x)$ is compactly supported or decays to zero sufficiently rapidly as $x \to \pm\infty$.

(a) Show that the Greens function $G(x, \xi)$ satisfying

$$\begin{aligned} &-\frac{d^2G}{dx^2} + G = \delta(x-\xi) \qquad -\infty < x < \infty, \\ &G(x, \xi) \to 0 \qquad \text{as } x \to \pm\infty \end{aligned}$$

is given by

$$G(x, \xi) = \frac{1}{2} e^{-|x-\xi|}.$$

(b) Write down the Green's function representation of the solution of (2). Verify explicitly that it is a solution.