SOLUTIONS: PROBLEM SET 4
Math 207B, Winter 2012

1. (a) Consider the 2π-periodic function $f(x;\epsilon)$ defined for $\epsilon > 0$ by

$$f(x;\epsilon) = \begin{cases} 1/\epsilon & \text{if } 0 < x < \epsilon, \\ 0 & \text{if } \epsilon < x < 2\pi. \end{cases}, \qquad f(x+2\pi;\epsilon) = f(x;\epsilon).$$

Sketch the graph of $f(x,\epsilon)$ on the real line. Compute its Fourier series

$$f(x;\epsilon) = \sum_{n=-\infty}^{\infty} f_n(\epsilon) e^{inx}.$$

(b) Define the 2π-periodic δ-function δ_p, or 'δ-comb', by

$$\delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n)$$

where $\delta(x)$ is δ-function on the real line supported at $x = 0$. Draw a picture of δ_p. Show that a formal computation of the Fourier coefficients of δ_p gives

$$\delta_p(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx}.$$

(c) Show that you recover the Fourier series in (b) by taking the limit as $\epsilon \to 0^+$ of the Fourier series in (a).

Solution

- (a) For $n \neq 0$, we have

$$\begin{aligned} f_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x;\epsilon) e^{-inx}\, dx \\ &= \frac{1}{2\pi\epsilon} \int_0^{\epsilon} e^{-inx}\, dx \\ &= -\frac{1}{2\pi\epsilon i n} \left[e^{-inx} \right]_0^{\epsilon} \\ &= \frac{1}{2\pi\epsilon i n} \left(1 - e^{-in\epsilon} \right) \\ &= \frac{1}{\pi\epsilon n} \sin\left(\frac{n\epsilon}{2} \right) e^{-in\epsilon/2}. \end{aligned}$$

Similarly, for $n = 0$, we have

$$f_0 = \frac{1}{2\pi\epsilon} \int_0^\epsilon dx = \frac{1}{2\pi}.$$

- (b) We integrate over the period $-\pi < x < \pi$, so that the singularities in the δ-functions are not at one of the endpoints. There is only one singularity at $x = 0$ in this period, so

$$\delta_{pn} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x) e^{-inx}\, dx = \frac{1}{2\pi}.$$

- (c) Since $\sin\theta/\theta \to 1$ as $\theta \to 0$, we have

$$\lim_{\epsilon \to 0} \left[\frac{1}{\pi \epsilon n} \sin\left(\frac{n\epsilon}{2}\right) e^{-in\epsilon/2} \right] = \lim_{\epsilon \to 0} \left[\frac{1}{2\pi} \frac{\sin(n\epsilon/2)}{n\epsilon/2} e^{-in\epsilon/2} \right] = \frac{1}{2\pi}$$

and we recover the result of (b).

2. Consider the 2π-periodic square wave S defined by

$$S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ 0 & \text{if } -\pi < x < 0. \end{cases}, \qquad S(x + 2\pi) = S(x).$$

(a) Sketch the graph of S on the real line and explain why the (distributional) derivative of S is given by

$$S'(x) = \delta_p(x) - \delta_p(x - \pi)$$

where δ_p is the periodic δ-function from Problem 1.
(b) Compute the Fourier series of S

$$S(x) = \sum_{n=-\infty}^{\infty} S_n e^{inx}.$$

(c) Show that the formal term-by-term derivative of this series agrees with the Fourier series of $\delta_p(x) - \delta_p(x - \pi)$ from Problem 1.

Solution

- (a) The function $S(x)$ is constant for $x \neq n\pi$, which differentiates to zero. It has jump discontinuities of size 1 at even multiples of π, which contribute $\delta_p(x)$ to the derivative, and jump discontinuities of size -1 at odd multiples of π, which contribute $-\delta_p(x - \pi)$ to the derivative.

- (b) For $n \neq 0$, we have

$$\begin{aligned} S_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S(x) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \int_{0}^{\pi} e^{-inx}\, dx \\ &= \frac{1}{2\pi} \left[\frac{e^{-inx}}{-in} \right]_0^{\pi} \\ &= \frac{1}{2\pi i n} \left(1 - e^{-in\pi} \right) \\ &= \frac{1}{2\pi i n} \left[1 - (-1)^n \right]. \end{aligned}$$

For $n = 0$, we have

$$S_0 = \frac{1}{2\pi}\int_0^\pi dx = \frac{1}{2}.$$

Thus,

$$S(x) = \frac{1}{2} + \sum_{n\neq 0} \frac{[1-(-1)^n]}{2\pi i n} e^{inx}.$$

(The series converges pointwise to $S(x)$ except where it has a jump discontinuity, where it converges to the average value of $1/2$. The convergence is not uniform, however, and we get the Gibb's phenomenon at the jump discontinuities of $S(x)$.)

- (c) Formal term-by-term differentiation gives

$$\begin{aligned}
S'(x) &= \frac{1}{2\pi}\sum_{n\neq 0}[1-(-1)^n]\,e^{inx} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx} - \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n e^{inx} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx} - \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(x-\pi)} \\
&= \delta_p(x) - \delta_p(x-\pi).
\end{aligned}$$

(The series can be understood as converging in the sense of distributions.)

3. Consider the non-homogeneous Sturm-Liouville problem

$$\begin{aligned} -u'' &= \lambda u + f(x) \qquad 0 < x < 1, \\ u'(0) &= 0, \qquad u(1) = 0 \end{aligned} \tag{1}$$

where $\lambda = k^2$ with $k > 0$ is a strictly positive constant.

(a) Give the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the homogeneous problem, $n = 1, 2, 3, \dots$, which satisfy

$$\begin{aligned} -\phi_n'' &= \lambda_n \phi_n \qquad 0 < x < 1, \\ \phi_n'(0) &= 0, \qquad \phi_n(1) = 0. \end{aligned}$$

(b) Write out the eigenfunction expansion for the Green's function $G(x, \xi; \lambda)$.

(b) Find an explicit expression for the Green's function $G(x, \xi; \lambda)$ in terms of appropriate solutions of the homogeneous equation. Show that your solution has poles at the eigenvalues $\lambda = \lambda_n$ and that its residues give the eigenfunctions.

Solution

- (a) The eigenvalues and orthonormal eigenfunctions are

$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\pi\right]^2, \qquad \phi_n(x) = \sqrt{2}\cos\left[\left(n - \frac{1}{2}\right)\pi x\right]$$

for $n = 1, 2, 3, \dots$.

- The bilinear eigenfunction expansion of the Green's function $G(x, \xi; \lambda)$ is

$$G(x, \xi; \lambda) = \sum_{n=1}^{\infty} \frac{2\cos\left[(n - 1/2)\,\pi x\right]\cos\left[(n - 1/2)\,\pi \xi\right]}{\left[(n - 1/2)\,\pi\right]^2 - \lambda}. \tag{2}$$

(By the Weierstrass M-test, this series converges uniformly for any $\lambda \in \mathbb{C}$ that is not one of the eigenvalues λ_n.)

- (b) The Green's function satisfies

$$\begin{aligned} -\frac{d^2G}{dx^2} &= \lambda G + \delta(x - \xi) \qquad 0 < x < 1, \\ \frac{dG}{dx}(0, \xi; \lambda) &= 0, \qquad G(1, \xi; \lambda) = 0. \end{aligned}$$

We will assume that $\lambda \neq 0$. (See below for the case $\lambda = 0$.)

- For $0 < x < \xi$, we have

$$-\frac{d^2G}{dx^2} = \lambda G, \qquad \frac{dG}{dx}(0, \xi; \lambda) = 0,$$

whose solution is

$$G(x, \xi; \lambda) = A(\xi; \lambda) \cos\left(\sqrt{\lambda} x\right)$$

where A is a function of integration. (Here, we may choose either value of the square-root.)

- For $\xi < x < 1$, we have

$$-\frac{d^2G}{dx^2} = \lambda G, \qquad G(1, \xi; \lambda) = 0,$$

whose solution is

$$G(x, \xi; \lambda) = B(\xi; \lambda) \sin\left(\sqrt{\lambda}(1 - x)\right).$$

- To ensure that the Green's function is continuous at $x = \xi$, we choose

$$A(\xi; \lambda) = C(\lambda) \sin\left(\sqrt{\lambda}(1 - \xi)\right), \qquad B(\xi; \lambda) = C(\lambda) \cos\left(\sqrt{\lambda}\xi\right),$$

so

$$\begin{aligned} G(x, \xi; \lambda) &= \begin{cases} C(\lambda) \cos\left(\sqrt{\lambda} x\right) \sin\left(\sqrt{\lambda}(1 - \xi)\right) & \text{if } 0 < x < \xi, \\ C(\lambda) \cos\left(\sqrt{\lambda}\xi\right) \sin\left(\sqrt{\lambda}(1 - x)\right) & \text{if } \xi < x < 1. \end{cases} \\ &= C(\lambda) \cos\left(\sqrt{\lambda} x_<\right) \sin\left(\sqrt{\lambda}(1 - x_>)\right) \end{aligned}$$

where

$$x_< = \min(x, \xi), \qquad x_> = \max(x, \xi).$$

- The requirement that dG/dx jumps by -1 across $x = \xi$ implies that

$$\begin{aligned} &-\sqrt{\lambda} C(\lambda) \cos\left(\sqrt{\lambda}\xi\right) \cos\left(\sqrt{\lambda}(1 - \xi)\right) \\ &\quad + \sqrt{\lambda} C(\lambda) \sin\left(\sqrt{\lambda}\xi\right) \sin\left(\sqrt{\lambda}(1 - \xi)\right) = -1 \end{aligned}$$

or, by use of the addition formula for cosine,

$$\sqrt{\lambda} C(\lambda) \cos \sqrt{\lambda} = 1.$$

Hence

$$C(\lambda) = \frac{1}{\sqrt{\lambda} \cos \sqrt{\lambda}}$$

and

$$G(x, \xi; \lambda) = \frac{\cos\left(\sqrt{\lambda} x_<\right) \sin\left(\sqrt{\lambda}(1 - x_>)\right)}{\sqrt{\lambda} \cos \sqrt{\lambda}}.$$

- Note that $\sin z/z$ and $\cos z$ are analytic functions of z^2 (their Taylor series involve only even powers of z) so G depends on λ, not $\sqrt{\lambda}$, and does not have a branch cut. Furthermore, G is an analytic function of λ unless the denominator $\sqrt{\lambda} \cos \sqrt{\lambda}$ vanishes.

- If $\cos \sqrt{\lambda} = 0$, then $\sqrt{\lambda} = \pm(n - 1/2)\pi$ and $\lambda = \lambda_n$ is an eigenvalue. We show below that G has simple poles at these values of λ.

- The point $\lambda = 0$ is a removable singularity of G. Since $\sin z/z \to 1$ as $z \to 0$, we have

$$\lim_{\lambda \to 0} G(x, \xi; \lambda) = 1 - x_> = \begin{cases} 1 - \xi & \text{if } 0 < x < \xi, \\ 1 - x & \text{if } \xi < x < 1. \end{cases}$$

  One can verify explicitly that this limit is the Green's function $G(x, \xi; 0)$. Thus, $G(x, \xi; \lambda)$ is analytic at $\lambda = 0$.

- The residue of G at the simple pole $\lambda = \lambda_n$ is given by

$$\begin{aligned} \operatorname*{Res}_{\lambda=\lambda_n} G(x, \xi; \lambda) &= \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\, G(x, \xi; \lambda) \\ &= \frac{1}{\sqrt{\lambda_n}} \cos\left(\sqrt{\lambda_n} x_<\right) \sin\left(\sqrt{\lambda_n}(1 - x_>)\right) \lim_{\lambda \to \lambda_n} \left[\frac{\lambda - \lambda_n}{\cos \sqrt{\lambda}}\right]. \end{aligned}$$

By l'Hôspital's rule, or Taylor expansion,

$$\begin{aligned}\lim_{\lambda\to\lambda_n}\left[\frac{\lambda-\lambda_n}{\cos\sqrt{\lambda}}\right] &= \lim_{\lambda\to\lambda_n}\left[\frac{1}{d/d\lambda(\cos\sqrt{\lambda})}\right]\\ &= -\lim_{\lambda\to\lambda_n}\left[\frac{2\sqrt{\lambda}}{\sin\sqrt{\lambda}}\right]\\ &= -\frac{2\sqrt{\lambda_n}}{\sin\sqrt{\lambda_n}}.\end{aligned}$$

Hence,

$$\operatorname*{Res}_{\lambda=\lambda_n} G(x,\xi;\lambda) = -\frac{2\cos\left(\sqrt{\lambda_n}x_<\right)\sin\left(\sqrt{\lambda_n}(1-x_>)\right)}{\sin\sqrt{\lambda_n}}.$$

We have

$$\begin{aligned}\sin\left(\sqrt{\lambda_n}(1-x_>)\right) &= \sin\left(\sqrt{\lambda_n}\right)\cos\left(\sqrt{\lambda_n}x_>\right)\\ &\quad -\cos\left(\sqrt{\lambda_n}\right)\sin\left(\sqrt{\lambda_n}x_>\right)\\ &= \sin\left(\sqrt{\lambda_n}\right)\cos\left(\sqrt{\lambda_n}x_>\right)\end{aligned}$$

so

$$\operatorname*{Res}_{\lambda=\lambda_n} G(x,\xi;\lambda) = -2\cos\left(\sqrt{\lambda_n}x_<\right)\cos\left(\sqrt{\lambda_n}x_>\right). \tag{3}$$

- This result agrees with (2), which gives

$$\operatorname*{Res}_{\lambda=\lambda_n} G(x,\xi;\lambda) = -2\cos\left[(n-1/2)\,\pi x\right]\cos\left[(n-1/2)\,\pi\xi\right].$$

In fact, we can read off the eigenfunctions $\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right)$ from the expression (3) for the residue of the Green's function $G(x,\xi;\lambda)$ at λ_n.

4. Consider the non-homogeneous Sturm-Liouville problem on the real line

$$\begin{aligned} &-u'' + u = f(x) \qquad -\infty < x < \infty, \\ &u(x) \to 0 \qquad \text{as } x \to \pm\infty \end{aligned} \tag{4}$$

where $f(x)$ is compactly supported or decays to zero sufficiently rapidly as $x \to \pm\infty$.

(a) Show that the Greens function $G(x, \xi)$ satisfying

$$\begin{aligned} &-\frac{d^2G}{dx^2} + G = \delta(x - \xi) \qquad -\infty < x < \infty, \\ &G(x, \xi) \to 0 \qquad \text{as } x \to \pm\infty \end{aligned}$$

is given by

$$G(x, \xi) = \frac{1}{2} e^{-|x-\xi|}.$$

(b) Write down the Green's function representation of the solution of (4). Verify explicitly that it is a solution.

Solution

- (a) For $x \neq \xi$, the Green's function satisfies the homogeneous equation and the BC at the appropriate endpoint:

$$\begin{aligned} &-\frac{d^2G}{dx^2} + G = 0 \qquad -\infty < x < \xi, \qquad G(x, \xi) \to 0 \quad \text{as } x \to -\infty, \\ &-\frac{d^2G}{dx^2} + G = 0 \qquad \xi < x < \infty, \qquad G(x, \xi) \to 0 \quad \text{as } x \to \infty. \end{aligned}$$

  It follows that

$$G(x, \xi) = \begin{cases} A(\xi)e^{x} & \text{if } -\infty < x < \xi, \\ B(\xi)e^{-x} & \text{if } \xi < x < \infty. \end{cases}$$

  where A, B are functions of integration.

- To ensure that $G(x, \xi)$ is continuous at $x = \xi$, we choose

$$A(\xi) = ce^{-\xi}, \qquad B(\xi) = ce^{\xi}$$

  where c is a constant (as follows from the general theory).

- In order to obtain the δ-function supported at ξ, we require that dG/dx satisfies the jump condition

$$\left[\frac{dG}{dx}\right]_{x=\xi} = -1$$

where

$$\begin{aligned}\left[\frac{dG}{dx}\right]_{x=\xi} &= \lim_{x\to\xi^+}\frac{dG}{dx}(x,\xi) - \lim_{x\to\xi^-}\frac{dG}{dx}(x,\xi) \\ &= -B(\xi)e^{-\xi} - A(\xi)e^{\xi} \\ &= -2c.\end{aligned}$$

Hence, $c = 1/2$ and

$$\begin{aligned}G(x,\xi) &= \frac{1}{2}\exp(x_<)\exp(-x_>) \\ &= \begin{cases}\frac{1}{2}e^{x-\xi} & \text{if } -\infty < x < \xi, \\ \frac{1}{2}e^{\xi-x} & \text{if } \xi < x < \infty\end{cases} \\ &= \frac{1}{2}e^{-|x-\xi|}.\end{aligned}$$

- (b) The Green's function representation of the solution is

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|} f(\xi)\, d\xi.$$

- Splitting up the integration interval, we can write this solution as

$$\begin{aligned}u(x) &= \frac{1}{2}\int_{-\infty}^{x} e^{-(x-\xi)} f(\xi)\, d\xi + \frac{1}{2}\int_{x}^{\infty} e^{x-\xi} f(\xi)\, d\xi \\ &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\, d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi} f(\xi)\, d\xi.\end{aligned}$$

We have

$$\begin{aligned}\frac{d}{dx}\left[e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\, d\xi\right] &= e^{-x}\cdot e^{x} f(x) - e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\, d\xi \\ &= f(x) - e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\, d\xi\end{aligned}$$

and

$$\frac{d^2}{dx^2}\left[e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\,d\xi\right] = f'(x) - e^{-x}\cdot e^{x} f(x) + e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\,d\xi$$
$$= f'(x) - f(x) + e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\,d\xi.$$

Similarly

$$\frac{d^2}{dx^2}\left[e^{x}\int_{x}^{\infty} e^{-\xi} f(\xi)\,d\xi\right] = -f'(x) - f(x) + e^{x}\int_{x}^{\infty} e^{-\xi} f(\xi)\,d\xi.$$

Adding these expressions, we get

$$u'' = -f(x) + \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} f(\xi)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi} f(\xi)\,d\xi = -f(x) + u,$$

which verifies explicitly that $-u'' + u = f(x)$.

- To verify the boundary conditions, assume for simplicity that $f(x)$ is a continuous function with compact support, meaning that it vanishes outside a finite interval $[-a, a]$. Then for $x < a$

$$\int_{-\infty}^{x} e^{\xi} f(\xi)\,d\xi = 0, \quad \int_{x}^{\infty} e^{-\xi} f(\xi)\,d\xi = A, \quad A = \int_{-a}^{a} e^{-\xi} f(\xi)\,d\xi$$

so $u(x) = (A/2)e^{x}$, and for $x > a$

$$\int_{-\infty}^{x} e^{\xi} f(\xi)\,d\xi = Be^{-x}, \quad \int_{x}^{\infty} e^{-\xi} f(\xi)\,d\xi = 0, \quad B = \int_{-a}^{a} e^{\xi} f(\xi)\,d\xi$$

so $u(x) = (B/2)e^{-x}$. Hence, $u(x) \to 0$ as $x \to \pm\infty$.