PROBLEM SET 5
Math 207B, Winter 2012
Due: Fri., Feb. 17

1. Consider the non-homogeneous regular Sturm-Liouville problem

$$\begin{aligned} &-[p(x)u']' + q(x)u = \lambda u + f(x), \qquad a < x < b, \\ &u(a) = 0, \qquad u(b) = 0, \end{aligned} \tag{1}$$

with eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$, and (real-valued) orthonormal eigenfunctions $\{\phi_1, \phi_2, \dots, \phi_n, \dots\}$. Let $P_n : L^2(a,b) \to L^2(a,b)$ denote the projection onto the orthogonal complement of the nth eigenspace, defined by

$$P_n f(x) = f(x) - (f, \phi_n)\, \phi_n(x).$$

(a) Suppose that $\lambda = \lambda_n$ in (1). If $(f, \phi_n) = 0$, show that a solution of (1) is given by

$$\begin{aligned} u(x) &= \int_a^b G(x, \xi; \lambda_n) f(\xi)\, d\xi, \\ G(x, \xi; \lambda_n) &= \sum_{k \neq n} \frac{\phi_k(x)\phi_k(\xi)}{\lambda_k - \lambda_n}. \end{aligned} \tag{2}$$

Is this solution unique?

(b) For arbitrary $f \in L^2(a,b)$ show that $u(x)$ defined in (2) is the unique solution of the following problem:

$$\begin{aligned} &-[p(x)u']' + q(x)u = \lambda_n u + P_n f(x), \quad a < x < b, \\ &u(a) = 0, \quad u(b) = 0, \qquad P_n u = u. \end{aligned}$$

(The function G is called the modified Green's function.)

(c) Compute the modified Green's function $G(x, \xi; \lambda_n)$ for the problem

$$\begin{aligned} &-u'' = \lambda_n u + f(x), \quad 0 < x < 1, \\ &u'(0) = 0, \qquad u'(1) = 0 \end{aligned}$$

in terms of solutions of the homogeneous equation, where $\lambda_n = n^2\pi^2$ for $n = 0, 1, 2, \dots$.

2. Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_a^b \left[\frac{1}{2}(u'')^2 + \frac{1}{2}qu^2 - fu \right] dx$$

where $f, q : [a,b] \to \mathbb{R}$ are continuous functions and $X = H_0^2(a,b)$ is the Sobolev space

$$X = \left\{ u : u, u', u'' \in L^2(a,b), u(a) = u'(a) = u(b) = u'(b) = 0 \right\}$$

Derive the Euler-Lagrange equation satisfied by a smooth minimizer $u \in C^4[a,b]$ of J. What is the variational derivative $\delta J/\delta u$ of J?

3. Consider the regular weighted Sturm-Liouville eigenvalue problem

$$\begin{aligned} &-(pu')' + qu = \lambda r u \qquad a < x < b, \\ &u(a) = 0, \qquad u(b) = 0 \end{aligned}$$

where $p(x)$, $q(x)$, $r(x)$ are smooth real-valued functions and $p, r > 0$.

(a) If the eigenvalues are $\lambda_1 < \lambda_2 < \ldots, \lambda_n < \ldots$, show that the minimum eigenvalue λ_1 is given by

$$\lambda_1 = \min_{u \in H_0^1(a,b)} R(u), \qquad R(u) = \frac{\int_a^b \left[p(u')^2 + qu^2\right] dx}{\int_a^b ru^2\, dx}.$$

(b) Consider the (singular) Sturm-Liouville problem

$$\begin{aligned} -(xu')' &= \lambda x u \qquad 0 < x < 1 \\ xu' \to 0 \text{ as } x &\to 0, \qquad u(1) = 0 \end{aligned}$$

with Rayleigh quotient

$$R(u) = \frac{\int_0^1 x(u')^2\, dx}{\int_0^1 xu^2\, dx}.$$

Assuming the result in (a) holds for this singular problem, use a trial function of the form

$$u(x) = (1-x)(ax+b)$$

to obtain an upper bound for the minimum eigenvalue λ_1. Compare your answer with the numerical result $\lambda_1 = 5.783\ldots$.

Remark. After a rescaling, this ODE is the zeroth-order Bessel equation. The eigenfunctions are $J_0(\sqrt{\lambda_n}x)$ where $J_0(z)$ is the Bessel function of order zero and $\sqrt{\lambda_n} = j_{0,n}$ is the nth zero of $J_0(z)$. The first zero is $j_{0,1} \approx 2.4048$.