Solutions: Problem set 5
Math 207B, Winter 2012

1. Consider the non-homogeneous regular Sturm-Liouville problem

$$-[p(x)u']' + q(x)u = \lambda u + f(x), \quad a < x < b, \qquad u(a) = u(b) = 0, \tag{1}$$

with eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$, and (real-valued) orthonormal eigenfunctions $\{\phi_1, \phi_2, \dots, \phi_n, \dots\}$. Let $P_n : L^2(a, b) \to L^2(a, b)$ denote the projection onto the orthogonal complement of the nth eigenspace, defined by

$$P_n f(x) = f(x) - (f, \phi_n)\, \phi_n(x).$$

(a) Suppose that $\lambda = \lambda_n$ in (1). If $(f, \phi_n) = 0$, show that a solution of (1) is given by

$$u(x) = \int_a^b G(x, \xi; \lambda_n) f(\xi)\, d\xi, \quad G(x, \xi; \lambda_n) = \sum_{k \neq n} \frac{\phi_k(x)\phi_k(\xi)}{\lambda_k - \lambda_n}. \tag{2}$$

Is this solution unique? (G is called the modified Green's function.)
(b) For arbitrary $f \in L^2(a, b)$ show that $u(x)$ defined in (2) is the unique solution of the following problem:

$$\begin{aligned} &-[p(x)u']' + q(x)u = \lambda_n u + P_n f(x), \quad a < x < b, \\ &u(a) = 0, \quad u(b) = 0, \qquad P_n u = u. \end{aligned}$$

(c) Compute the modified Green's function $G(x, \xi; \lambda_n)$ for the problem

$$\begin{aligned} &- u'' = \lambda_n u + f(x) - 2\left(\int_0^1 f(\xi)\cos(n\pi\xi\, d\xi\right)\cos(n\pi x), \quad 0 < x < 1, \\ &u'(0) = 0, \qquad u'(1) = 0, \qquad \int_0^1 u(x)\cos(n\pi x)\, d\xi = 0 \end{aligned} \tag{3}$$

in terms of solutions of the homogeneous equation, where $\lambda_n = n^2\pi^2$ for $n = 0, 1, 2, \dots$.

Solution

- (a) Write (1) as $(L - \lambda I)u = f$ where
$$L = -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x).$$
Expanding u, f as
$$u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x), \qquad f(x) = \sum_{k=1}^{\infty} f_k \phi_k(x),$$
where
$$f_k = (f, \phi_k) = \int_a^b f(x)\phi_k(x)\,dx,$$
and using $L\phi_k = \lambda_k \phi_k$, we get that
$$(L - \lambda I)u = \sum_{k=1}^{\infty} (\lambda_k - \lambda)\, c_k \phi_k(x).$$
Hence, $(L - \lambda_n I)u = f$ if and only if
$$(\lambda_k - \lambda_n)\, c_k = f_k \qquad \text{for } k = 1, 2, 3, \dots.$$

- A solution for u exists if and only if $f_n = 0$, or $(f, \phi_n) = 0$. In that case, the general solution is
$$u(x) = \sum_{k \neq n} \frac{f_k}{\lambda_k - \lambda_n}\, \phi_k(x) + c\phi_n(x)$$
where c is an arbitrary constant. If a solution exists it is not unique (we can add an arbitrary solution of the homogeneous equation) but there is a unique solution u such that $(u, \phi_n) = 0$, which implies that $c = 0$.

- If $c = 0$, we can write this solution as
$$\begin{aligned} u(x) &= \sum_{k \neq n} \frac{f_k}{\lambda_k - \lambda_n}\, \phi_k(x) \\ &= \sum_{k \neq n} \frac{1}{\lambda_k - \lambda_n}\left[\int_a^b f(\xi)\phi_k(\xi)\,d\xi\right]\phi_k(x) \\ &= \int_a^b \left[\sum_{k \neq n} \frac{\phi_k(x)\phi_k(\xi)}{\lambda_k - \lambda_n}\right] f(\xi)\,d\xi, \end{aligned}$$

which gives the Green's function representation in (2).

- (b) This follows immediately from the discussion above. The operator $(L - \lambda_n I)^+$, given by

$$(L - \lambda_n I)^+ f(x) = \int_a^b G(x, \xi; \lambda_n) f(\xi)\, d\xi,$$

in which we project both f and u onto the subspace orthogonal to the eigenfunction provides a generalization of the inverse to a singular problem. It is sometime referred to as a pseudoinverse, or Moore-Penrose pseudoinverse. Note that G is a symmetric function of (x, ξ), so $(L - \lambda_n I)^+$ is self-adjoint.

- (c) The eigenvalues of this problem are

$$\lambda_n = n^2\pi^2, \qquad n = 0, 1, 2, 3, \ldots$$

with orthonormal eigenfunctions

$$\phi_0(x) = 1, \quad \phi_n(x) = \sqrt{2}\cos(n\pi x) \quad = 1, 2, 3, \ldots.$$

- The modified Green's function $G(x, \xi; \lambda_n)$, with eigenfunction expansion (2), satisfies

$$\begin{aligned} &-\frac{d^2G}{d^2x} = \lambda_n G + P_n \delta(x - \xi), \quad 0 < x < 1, \\ &\frac{dG}{dx}(0, \xi; \lambda_n) = 0, \qquad \frac{dG}{dx}(1, \xi; \lambda_n) = 0, \\ &\int_0^1 G(x, \xi; \lambda_n)\phi_n(x)\, dx = 0. \end{aligned}$$

- Consider first the case $n = 0$, with $\lambda_0 = 0$ and $\phi_0 = 1$. Then

$$\begin{aligned} P_0\delta(x - \xi) &= \delta(x - \xi) - \left(\int_0^1 \delta(x - \xi) \cdot 1\, dx\right) \cdot 1 \\ &= \delta(x - \xi) - 1, \end{aligned}$$

and $G(x,\xi;0)$ satisfies

$$-\frac{d^2G}{d^2x} = \delta(x-\xi) - 1, \quad 0 < x < 1,$$
$$\frac{dG}{dx}(0,\xi;0) = 0, \qquad \frac{dG}{dx}(1,\xi;0) = 0,$$
$$\int_0^1 G(x,\xi;0)\,dx = 0.$$

- For $0 < x < \xi$, we have

$$-\frac{d^2G}{d^2x} = -1, \qquad \frac{dG}{dx}(0,\xi;0) = 0,$$

whose general solution is

$$G(x,\xi;0) = \frac{1}{2}x^2 + A(\xi)$$

where $A(\xi)$ is a function of integration.

- For $\xi < x < 1$, we have

$$-\frac{d^2G}{d^2x} = -1, \qquad \frac{dG}{dx}(1,\xi;0) = 0,$$

whose general solution is

$$G(x,\xi;0) = \frac{1}{2}(1-x)^2 + B(\xi).$$

- The continuity of G at $x = \xi$ implies that

$$A(\xi) = \frac{1}{2}(1-\xi)^2 + C(\xi), \qquad B(\xi) = \frac{1}{2}\xi^2 + C(\xi)$$

where $C(\xi)$ is an arbitrary function of ξ. Hence

$$G(x,\xi;0) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}(1-\xi)^2 + C(\xi) & \text{if } 0 < x < \xi, \\ \frac{1}{2}\xi^2 + \frac{1}{2}(1-x)^2 + C(\xi) & \text{if } \xi < x < 1. \end{cases}$$

- Note that G automatically satisfies the correct jump condition

$$\left[\frac{dG}{dx}\right]_{x=\xi} = -(1-\xi) - \xi = -1,$$

which is a consequence of projecting the δ-function onto the range of solvable right-hand sides.

- To determine the arbitrary function $C(\xi)$, we impose the orthogonality condition $\int_0^1 G\,dx = 0$. We have

$$\begin{aligned}
&\int_0^1 G(x,\xi;0)\,dx \\
&= \frac{1}{2}\int_0^{\xi} \left[x^2 + (1-\xi)^2\right]\,dx + \frac{1}{2}\int_{\xi}^1 \left[\xi^2 + (1-x)^2\right]\,dx + C(\xi) \\
&= \frac{1}{2}\left[\frac{1}{3}x^3 + x(1-\xi)^2\right]_0^{\xi} - \frac{1}{2}\left[(1-x)\xi^2 + \frac{1}{3}(1-x)^3\right]_{\xi}^1 + C(\xi) \\
&= \frac{1}{6}\xi^3 + \frac{1}{2}\xi(1-\xi)^2 + \frac{1}{2}(1-\xi)\xi^2 + \frac{1}{6}(1-\xi)^3 + C(\xi) \\
&= \frac{1}{6} + C(\xi),
\end{aligned}$$

so we take $C = -1/6$. The modified Green's function is therefore given by

$$\begin{aligned}
G(x,\xi;0) &= \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}(1-\xi)^2 - \frac{1}{6} & \text{if } 0 < x < \xi, \\ \frac{1}{2}\xi^2 + \frac{1}{2}(1-x)^2 - \frac{1}{6} & \text{if } \xi < x < 1. \end{cases} \\
&= \frac{1}{2}\left[x_<^2 + (1-x_>)^2 - \frac{1}{3}\right].
\end{aligned}$$

- The case $n \geq 1$, with

$$\lambda_n = n^2\pi^2, \qquad \phi_n(x) = \sqrt{2}\cos(n\pi x)$$

is analogous to $n = 0$. We have

$$P_n\delta(x-\xi) = \delta(x-\xi) - 2\cos(n\pi x)\cos(n\pi\xi)$$

and $G(x,\xi;\lambda_n)$ satisfies

$$-\frac{d^2G}{d^2x} = n^2\pi^2 G + \delta(x-\xi) - 2\cos(n\pi x)\cos(n\pi\xi), \quad 0 < x < 1,$$
$$\frac{dG}{dx}(0,\xi;\lambda_n) = 0, \qquad \frac{dG}{dx}(1,\xi;\lambda_n) = 0,$$
$$\int_0^1 G(x,\xi;\lambda_n)\cos(n\pi x)\,dx = 0.$$

- For $0 < x < \xi$, we have

$$-\frac{d^2G}{d^2x} = n^2\pi^2 G - 2\cos(n\pi x)\cos(n\pi\xi), \qquad \frac{dG}{dx}(0,\xi;0) = 0,$$

and for $\xi < x < 1$, we have

$$-\frac{d^2G}{d^2x} = n^2\pi^2 G - 2\cos(n\pi x)\cos(n\pi\xi), \qquad \frac{dG}{dx}(1,\xi;0) = 0.$$

- A fundamental pair of solutions of the homogeneous ODE

$$-\frac{d^2G}{dx^2} = n^2\pi^2 G$$

is $\{\cos(n\pi x), \sin(n\pi x)\}$. A particular solution of the nonhomogeneous ODE

$$-\frac{d^2G}{d^2x} = n^2\pi^2 G - 2\cos(n\pi x)\cos(n\pi\xi),$$

which is forced by a resonant term proportional to a solution $\cos(n\pi x)$ of the homogeneous equation, is

$$\frac{1}{n\pi} x\sin(n\pi x)\cos(n\pi\xi).$$

Superposing this particular solution with a solution of the homogeneous equation, we find that the most general solution of these ODEs for G that satisfies the appropriate boundary conditions is

$$G(x,\xi;\lambda_n) = \frac{1}{n\pi} x\sin(n\pi x)\cos(n\pi\xi) + A(\xi)\cos(n\pi x) \qquad \text{if } 0 < x < \xi,$$
$$G(x,\xi;\lambda_n) = \frac{1}{n\pi} x\sin(n\pi x)\cos(n\pi\xi) - \frac{1}{n\pi}\sin(n\pi x)\cos(n\pi\xi) + B(\xi)\cos(n\pi x) \qquad \text{if } \xi < x < 1$$

where $A(\xi)$, $B(\xi)$ are arbitrary functions of integration.

- The continuity of G at $x = \xi$ implies that
$$B(\xi) - A(\xi) = \frac{1}{n\pi}\sin(n\pi\xi).$$
The jump condition
$$\left[\frac{dG}{dx}\right]_{x=\xi} = -1$$
is then satisfied automatically, as in the case $n = 0$.

- It follows that
$$\begin{aligned} G(x,\xi;\lambda_n) &= \frac{1}{n\pi}x\sin(n\pi x)\cos(n\pi\xi) - \frac{1}{n\pi}\cos(n\pi x)\sin(n\pi\xi) \\ &\quad + C(\xi)\cos(n\pi x) \qquad\qquad \text{if } 0 < x < \xi, \\ G(x,\xi;\lambda_n) &= \frac{1}{n\pi}x\sin(n\pi x)\cos(n\pi\xi) - \frac{1}{n\pi}\sin(n\pi x)\cos(n\pi\xi) \\ &\quad + C(\xi)\cos(n\pi x) \qquad\qquad \text{if } \xi < x < 1 \end{aligned}$$
where $C(\xi)$ is an arbitrary function.

- To determine C, we impose the orthogonality condition $\int_0^1 G\phi_n\,dx = 0$. After some algebra, we find that
$$\int_0^1 G(x,\xi;\lambda_n)\cos(n\pi x)\,dx = \frac{1}{2}C(\xi) - \frac{1}{2n\pi}\xi\sin(n\pi\xi) - \frac{1}{4n^2\pi^2}\cos(n\pi\xi).$$
It follows that
$$C(\xi) = \frac{1}{n\pi}\xi\sin(n\pi\xi) + \frac{1}{2n^2\pi^2}\cos(n\pi\xi).$$

- The modified Green's function is therefore given by
$$\begin{aligned} G(x,\xi;\lambda_n) &= \frac{1}{n\pi}x\sin(n\pi x)\cos(n\pi\xi) + \frac{1}{n\pi}\xi\cos(n\pi x)\sin(n\pi\xi) \\ &\quad - \frac{1}{n\pi}\cos(n\pi x)\sin(n\pi\xi) + \frac{1}{2n^2\pi^2}\cos(n\pi x)\cos(n\pi\xi) \end{aligned}$$
if $0 < x < \xi$, and
$$\begin{aligned} G(x,\xi;\lambda_n) &= \frac{1}{n\pi}x\sin(n\pi x)\cos(n\pi\xi) + \frac{1}{n\pi}\xi\cos(n\pi x)\sin(n\pi\xi) \\ &\quad - \frac{1}{n\pi}\sin(n\pi x)\cos(n\pi\xi) + \frac{1}{2n^2\pi^2}\cos(n\pi x)\cos(n\pi\xi) \end{aligned}$$
if $\xi < x < 1$.

- Equivalently,

$$\begin{aligned} G(x,\xi;\lambda_n) = {} & \frac{1}{n\pi} x_< \sin(n\pi x_<)\cos(n\pi x_>) + \frac{1}{n\pi} x_> \cos(n\pi x_<)\sin(n\pi x_>) \\ & - \frac{1}{n\pi}\cos(n\pi x_<)\sin(n\pi x_>) + \frac{1}{2n^2\pi^2}\cos(n\pi x_<)\cos(n\pi x_>). \end{aligned}$$

- The solution of (3) then has the Green's function representation

$$u(x) = \int_0^1 G(x,\xi;\lambda_n) f(\xi)\, d\xi.$$

2. Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_a^b \left[\frac{1}{2}(u'')^2 + \frac{1}{2}qu^2 - fu\right] dx$$

where $f, q : [a, b] \to \mathbb{R}$ are continuous functions and $X = H_0^2(a, b)$ is the Sobolev space

$$X = \left\{u : u, u', u'' \in L^2(a, b), u(a) = u'(a) = u(b) = u'(b) = 0\right\}$$

Derive the Euler-Lagrange equation satisfied by a smooth minimizer $u \in C^4[a, b]$ of J. What is the variational derivative $\delta J/\delta u$ of J?

Solution

- We compute the directional derivative of J at $u \in X$ in the direction $h \in X$:

$$\begin{aligned} \left.\frac{d}{dt}J(u+th)\right|_{t=0} &= \frac{d}{dt}\int_a^b \left[\frac{1}{2}(u''+th'')^2 + \frac{1}{2}q(u+th)^2 - f(u+th)\right] dx\Big|_{t=0} \\ &= \int_a^b (u''h'' + quh - fh)\, dx. \end{aligned}$$

- If $u \in C^4[a, b]$ is smooth, then we can integrate by parts in this expression to get

$$\left.\frac{d}{dt}J(u+th)\right|_{t=0} = \int_a^b (u'''' + qu - f)\, h\, dx.$$

  The boundary terms vanish since $h \in X$.

- The variational derivative $\delta J/\delta u$ is defined by the identity

$$\left.\frac{d}{dt}J(u+th)\right|_{t=0} = \int_a^b \frac{\delta J}{\delta u} h\, dx,$$

  so

$$\frac{\delta J}{\delta u} = u'''' + qu - f.$$

- If J attains a minimum at $u \in X$, then for any $h \in X$ the function $J(u + th)$ of $t \in \mathbb{R}$ attains a minimum at $t = 0$, so that

$$\frac{d}{dt} J(u + th)\bigg|_{t=0} = 0 \qquad \text{for every } h \in X.$$

  If $u \in C^4[a, b]$ is smooth, it follows that

$$\int_a^b \frac{\delta J}{\delta u} h \, dx = 0 \qquad \text{for every } h \in X,$$

  and the duBois-Reymond lemma implies that $\delta J/\delta u = 0$, or

$$u'''' + qu = f.$$

  This forth-order ODE is the Euler-Lagrange equation for the functional $J(u)$.

3. Consider the regular weighted Sturm-Liouville eigenvalue problem

$$\begin{aligned} -(pu')' + qu &= \lambda r u \qquad a < x < b, \\ u(a) = 0, &\qquad u(b) = 0 \end{aligned}$$

where $p(x)$, $q(x)$, $r(x)$ are smooth real-valued functions and $p, r > 0$.
(a) If the eigenvalues are $\lambda_1 < \lambda_2 < \ldots, \lambda_n < \ldots$, show that the minimum eigenvalue λ_1 is given by

$$\lambda_1 = \min_{u \in H_0^1(a,b)} R(u), \qquad R(u) = \frac{\int_a^b \left[p(u')^2 + qu^2\right]\, dx}{\int_a^b r u^2\, dx}.$$

(b) Consider the (singular) Sturm-Liouville problem

$$\begin{aligned} -(xu')' &= \lambda x u \qquad 0 < x < 1 \\ xu' \to 0 \text{ as } x \to 0, &\qquad u(1) = 0 \end{aligned}$$

with Rayleigh quotient

$$R(u) = \frac{\int_0^1 x(u')^2\, dx}{\int_0^1 x u^2\, dx}.$$

Assuming the result in (a) holds for this singular problem, use a trial function of the form

$$u(x) = (1 - x)(ax + b)$$

to obtain an upper bound for the minimum eigenvalue λ_1. Compare your answer with the numerical result $\lambda_1 = 5.783\ldots$.

Solution

- (a) For $n = 1, 2, \ldots$, let $\phi_n(x)$ denote (real-valued) eigenfunctions with eigenvalues λ_n such that

  $$-(p\phi_n')' + q\phi_n = \lambda_n r \phi_n.$$

  The eigenfunctions are orthogonal with respect to the weighted inner product

  $$(u, v)_r = \int_a^b ruv\, dx.$$

  We normalize them so that

  $$\int_a^b r\phi_n\phi_m\, dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

- These eigenfunctions form a complete orthonormal set in the weighted space $L^2_r(a,b)$. We expand u as

$$u(x) = \sum_{n=1} c_n \phi_n(x), \qquad c_n = \int_a^b r u \phi_n \, dx.$$

Then

$$\begin{aligned}
\int_a^b r u^2 &= \int_a^b r \left(\sum_{n=1}^{\infty} c_n \phi_n \right) \left(\sum_{m=1}^{\infty} c_m \phi_m \right) dx \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n c_m \int_a^b r \phi_n \phi_m \, dx \\
&= \sum_{n=1}^{\infty} c_n^2.
\end{aligned}$$

- Also, integrating by parts,

$$\begin{aligned}
\int_a^b \left[p (u')^2 + q u^2 \right] dx &= \int_a^b \left[-(p u')' u + q u^2 \right] dx \\
&= \int_a^b \left[-(p u')' + q u \right] u \, dx.
\end{aligned}$$

We have

$$-(p u')' + q u = \sum_{n=1} \lambda_n c_n r \phi_n(x),$$

so

$$\begin{aligned}
\int_a^b \left[p (u')^2 + q u^2 \right] dx &= \int_a^b r \left(\sum_{n=1}^{\infty} \lambda_n c_n r \phi_n \right) \left(\sum_{m=1}^{\infty} c_m \phi_m \right) dx \\
&= \sum_{n=1}^{\infty} \lambda_n c_n^2.
\end{aligned}$$

- It follows that the Rayleigh quotient is given by

$$R(u) = \frac{\sum_{n=1}^{\infty} \lambda_n c_n^2}{\sum_{n=1}^{\infty} c_n^2}.$$

This is minimized when $c_1 \neq 0$ and $c_n = 0$ for $n \geq 2$, and

$$\min R(u) = \lambda_1.$$

- (b) If $u(x) = (1-x)(ax+b)$, we compute that

$$\int_0^1 x(u')^2\,dx = \frac{1}{6}a^2 + \frac{1}{3}ab + \frac{1}{2}b^2,$$
$$\int_0^1 xu^2\,dx = \frac{1}{60}a^2 + \frac{1}{15}ab + \frac{1}{12}b^2.$$

  Hence, for this trial function, the Rayleigh quotient is

$$R(u) = \frac{\int_0^1 x(u')^2\,dx}{\int_0^1 xu^2\,dx} = 10\left[\frac{a^2+2ab+3b^2}{a^2+4ab+5b^2}\right].$$

  We have $R = 6$ for $a = 0$ and $R = 10$ for $b = 0$, which gives an upper bound $\lambda_1 \leq 6$.

- If $b \neq 0$, we can write $R = R(c)$ where $c = a/b$ and

$$R(c) = 10\left[\frac{c^2+2c+3}{c^2+4c+5}\right].$$

  (See Figure 1 for a plot of $R(c)$.) We have

$$R'(c) = 20\left[\frac{c^2+2c-1}{(c^2+4c+5)^2}\right],$$

  and $R'(c) = 0$ at $c = \pm\sqrt{2}-1$. The minimum of R is attained at $c = \sqrt{2}-1$, where

$$R(\sqrt{2}-1) = 10\left(2-\sqrt{2}\right).$$

  It follows that

$$\lambda_1 \leq 10\left(2-\sqrt{2}\right) \leq 5.86.$$

  We could obtain sharper upper bounds by using higher-dimensional spaces of trial functions *e.g.* $u(x) = (1-x)(ax^2+bx+c)$.

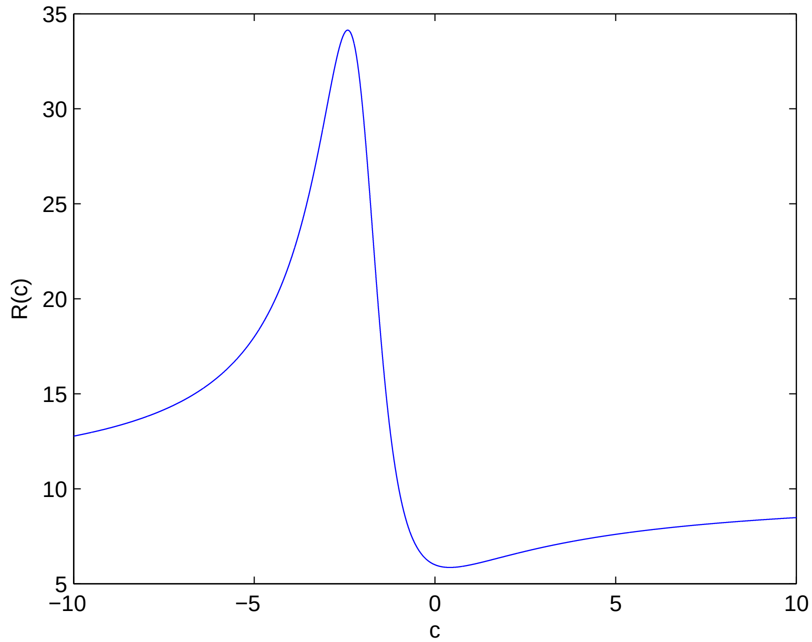

Figure 1: Plot of Rayleigh quotient R.