Problem Set 6

Math 207B, Winter 2012

Due: Fri., Feb. 24

1. Suppose that $\lambda \in \mathbb{C} \setminus [0, \infty)$ is not a nonnegative real number. Show that the Green's function for the BVP

$$-u'' = \lambda u + f(x), \qquad 0 < x < \infty, \qquad u(0) = 0, \quad u \in L^2(0, \infty)$$

is given by

$$G(x, \xi; \lambda) = \frac{1}{i\sqrt{-\lambda}} \sin\left(i\sqrt{-\lambda}x_<\right) \exp\left(-\sqrt{-\lambda}x_>\right)$$

where $\sqrt{-\lambda}$ is the branch of the square root with positive real part and

$$x_< = \min(x, \xi), \qquad x_> = \max(x, \xi).$$

What singularities does G have as a function of λ? Write down the Green function representation for the solution of the BVP.

2. Consider the Sturm-Liouville problem

$$-u'' = \lambda u \qquad 0 < x < \infty, \qquad (\cos\alpha)u(0) - (\sin\alpha)u'(0) = 0$$

where $0 \le \alpha \le \pi$ is a real constant. Show that this has an eigenfunction $u \in L^2(0, \infty)$ if and only if $-\pi/2 < \alpha < \pi$, and in that case $\lambda = -\cot^2\alpha$. (This problem also has a continuous spectrum with $0 \le \lambda < \infty$, similar to the one in Problem 1.)

3. Consider the Sturm-Liouville eigenvalue problem

$$-(x^2u')' = \lambda u \qquad 1 < x < e, \qquad u(1) = 0, \qquad u(e) = 0.$$

Is it regular or singular? Show that the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2\pi^2 + \frac{1}{4}, \qquad u_n(x) = x^{-1/2}\sin(n\pi\log x).$$

Write out the corresponding eigenfunction expansion of a function $f \in L^2(1, e)$.

Hint. Note that the ODE is an Euler equation and look for solutions of the form $u(x) = x^r$.

4. Consider the singular Sturm-Liouville eigenvalue problem for Legendre's equation

$$\begin{aligned} &-[(1-x^2)u']' = \lambda u \qquad -1 < x < 1, \\ &(1-x^2)u'(x) \to 0 \quad \text{as } x \to \pm 1. \end{aligned}$$

(a) Solve the ODE for $\lambda = 0$ and show that both endpoints $x = \pm 1$ are in the limit circle case.

(b) For $n = 0, 1, 2, \ldots$, define the Legendre polynomials $P_n(x)$ by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n \right].$$

(Note that P_n *is* a polynomial of degree n.) Show that the Legendre polynomials are eigenfunctions of the Legendre equation with eigenvalues

$$\lambda_n = n(n+1).$$

Hint. Let $v(x) = (x^2-1)^n$ and differentiate the equation $(x^2-1)v' = 2nxv$ $n+1$ times.

(c) With v as in (b), show that

$$\int_{-1}^{1} \left[\frac{d^n v}{dx^n} \right]^2 = (2n)! \int_{-1}^{1} (1-x)^n (1+x)^n \, dx = \frac{(n!)^2}{(2n)!} \int_{-1}^{1} (1+x)^{2n} \, dx$$

and deduce that

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}.$$

(d) Write out the orthogonality relations for the Legendre polynomials and the eigenfunction expansion of a function $f \in L^2(-1,1)$ with respect to the Legendre polynomials.