Solutions: Problem set 6
Math 207B, Winter 2012

1. Suppose that $\lambda \in \mathbb{C} \setminus [0, \infty)$ is not a nonnegative real number. Show that the Green's function for the BVP

$$-u'' = \lambda u + f(x), \qquad 0 < x < \infty, \qquad u(0) = 0, \quad u \in L^2(0, \infty)$$

is given by

$$G(x, \xi; \lambda) = \frac{1}{i\sqrt{-\lambda}} \sin\left(i\sqrt{-\lambda}x_<\right) \exp\left(-\sqrt{-\lambda}x_>\right)$$

where $\sqrt{-\lambda}$ is the branch of the square root with positive real part and

$$x_< = \min(x, \xi), \qquad x_> = \max(x, \xi).$$

What singularities does G have as a function of λ? Write down the Green function representation for the solution of the BVP.

Solution

- Up to a constant factor, the solution $u = u_1$ of the homogeneous equation $-u'' = \lambda u$ with $u(0) = 0$ is

$$u_1(x) = \sin\left(i\sqrt{-\lambda}x\right).$$

  Similarly, the solution $u = u_2$ of the homogeneous equation $-u'' = \lambda u$ with $u(x) \to 0$ as $x \to \infty$ is

$$u_2(x) = \exp\left(-\sqrt{-\lambda}x\right).$$

- The Green's function, which is symmetric since the problem is self-adjoint, is therefore given by

$$G(x, \xi; \lambda) = \begin{cases} c\sin\left(i\sqrt{-\lambda}x\right)\exp\left(-\sqrt{-\lambda}\xi\right) & \text{if } 0 < x < \xi, \\ c\sin\left(i\sqrt{-\lambda}\xi\right)\exp\left(-\sqrt{-\lambda}x\right) & \text{if } \xi < x < \infty \end{cases}$$

  where c is a constant.

- We choose c so that

$$\left[-\frac{dG}{dx}\right]_{x=\xi} = 1$$

which implies that

$$c\sqrt{-\lambda}\left[\sin\left(i\sqrt{-\lambda}\xi\right) + i\cos\left(i\sqrt{-\lambda}\xi\right)\right]\exp\left(-\sqrt{-\lambda}\xi\right) = 1.$$

From Euler's formula

$$\sin\left(i\sqrt{-\lambda}\xi\right) + i\cos\left(i\sqrt{-\lambda}\xi\right) = i\exp\left(\sqrt{-\lambda}\xi\right),$$

so $c = 1/(i\sqrt{-\lambda})$, which gives the result.

- The Green's function has a branch cut on the positive real axis $0 < \lambda < \infty$, across which $\sqrt{-\lambda}$ jumps from $0^+ + i\sqrt{\lambda}$ to $0^+ - i\sqrt{\lambda}$, and a branch point at $\lambda = 0$.

- The Green's function representation of the solution is

$$u(x) = \int_0^\infty G(x, \xi; \lambda) f(\xi)\, d\xi.$$

2. Consider the Sturm-Liouville problem

$$-u'' = \lambda u \qquad 0 < x < \infty, \qquad (\cos\alpha)u(0) - (\sin\alpha)u'(0) = 0$$

where $0 \leq \alpha \leq \pi$ is a real constant. Show that this has an eigenfunction $u \in L^2(0,\infty)$ if and only if $\pi/2 < \alpha < \pi$, and in that case $\lambda = -\cot^2\alpha$. (This problem also has a continuous spectrum with $0 \leq \lambda < \infty$, similar to the one in Problem 1.)

Solution

- The problem is self-adjoint so, from the general theory, any eigenvalue with eigenfunction $u \in L^2(0,\infty)$ must be real.

- If $\lambda = k^2 > 0$, then the general solution of the ODE is

  $$u(x) = c_1 \cos kx + c_2 \sin kx$$

  which is not square-integrable unless $u = 0$. Similarly, if $\lambda = 0$, then a solution $u(x) = c_1 + c_2 x$ is not square-integrable unless $u = 0$. Therefore there are no real eigenvalues with $\lambda \geq 0$. (These points do, however, belong to the continuous spectrum.)

- If $\lambda = -k^2 < 0$, where $k > 0$ without loss of generality, then

  $$u(x) = c_1 e^{-kx} + c_2 e^{kx}.$$

  For $k > 0$, this function is square-integrable if and only if $c_2 = 0$, and then $u(x) = e^{-kx}$ up to a constant factor. This satisfies the boundary condition at $x = 0$ if

  $$\cos\alpha + k\sin\alpha = 0$$

  or $k = -\cot\alpha$. Since $k > 0$ for the solution to decay, we need to have $\cot\alpha < 0$ or $\pi/2 < \alpha < \pi$. The corresponding eigenvalue and eigenfunction are

  $$\lambda = -\cot^2\alpha, \qquad u(x) = e^{(\cot\alpha)x}.$$

- Note that as the parameter α varies from $\pi/2$ to π, the eigenvalue $\lambda = 0$ detaches from the bottom of the continuous spectrum and then goes off to $-\infty$ as $\alpha \to \pi$. On the other hand, the absolutely continuous spectrum $0 \leq \lambda < \infty$ does not vary as we change the BC. In general, the absolutely continuous part of the spectrum of a Sturm-Liouville operator is independent of the boundary conditions, but the eigenvalues and the singular continuous part of the spectrum, if any exists, are not.

3. Consider the Sturm-Liouville eigenvalue problem

$$-(x^2u')' = \lambda u \qquad 1 < x < e, \qquad u(1) = 0, \qquad u(e) = 0.$$

Is it regular or singular? Show that the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2\pi^2 + \frac{1}{4}, \qquad u_n(x) = x^{-1/2}\sin(n\pi \log x).$$

Write out the corresponding eigenfunction expansion of a function $f \in L^2(1,e)$.

Solution

- The problem is regular since it is posed on a finite interval $1 \le x \le e$ and the coefficient function $p(x) = x^2$ is smooth and nonzero on the interval (including at the endpoints).

- Any eigenvalue is real and positive (*e.g.* from the Rayleigh quotient, since $p > 0$ and $q = 0$).

- The equation is a homogeneous Euler equation, so we look for solutions of the form $u(x) = x^r$. Then

$$-(x^2u')' = -(rx^{r+1})' = -r(r+1)x^r.$$

(Alternatively, the substitution $t = \log x$ reduces the ODE to a constant coefficient equation.) We therefore get a solution if $-r(r+1) = \lambda$ or

$$r^2 + r + \lambda = 0,$$

with roots

$$r = \frac{1}{2}\left(-1 \pm \sqrt{1-4\lambda}\right).$$

- If $\lambda < 1/4$, then $r = r_1, r_2$ where r_1, r_2 are distinct real exponents. The general solution of the ODE is

$$u(x) = c_1x^{r_1} + c_2x^{r_2}.$$

The boundary conditions imply that

$$c_1 + c_2 = 0, \qquad c_1e^{r_1} + c_2e^{r_2} = 0.$$

This linear system is non-singular for $r_1 \ne r_2$, so $c_1 = c_2 = 0$. It follows that $u = 0$ and λ is not an eigenvalue.

- If $\lambda = 1/4$, then we get a repeated root $r = -1/2$. A second linearly independent solution of the ODE is then given by $(\log x)x^r$, and the general solution is

$$u(x) = c_1 x^r + c_2 (\log x) x^r.$$

The boundary conditions implies that $c_1 = c_2 = 0$, so $\lambda = 1/4$ is not an eigenvalue.

- If $\lambda > 1/4$, then $r = -1/2 \pm i\omega$ where $\omega = \sqrt{\lambda - 1/4}$. We have

$$x^r = x^{-1/2} x^{\pm i\omega}, \quad x^{\pm i\omega} = e^{\pm i\omega \log x} = \cos(\omega \log x) \pm i \sin(\omega \log x),$$

so the general solution of the ODE is

$$u(x) = c_1 x^{-1/2} \cos(\omega \log x) + c_2 x^{-1/2} \sin(\omega \log x).$$

- The boundary condition at $x = 1$ implies that $c_1 = 0$. The boundary condition at $x = e$ is satisfied for $c_2 \neq 0$ if and only if $\sin \omega = 0$, or $\omega = n\pi$. It follows that the eigenvalues and eigenfunctions are

$$\lambda_n = n^2\pi^2 + \frac{1}{4}, \quad u_n(x) = x^{-1/2} \sin(n\pi \log x) \qquad n = 1, 2, 3, \ldots.$$

Note that, by using the substitution $t = \log x$, we get the orthogonality relation

$$\begin{aligned} \int_1^e u_n(x) u_m(x)\, dx &= \int_1^e x^{-1} \sin(n\pi \log x) \sin(m\pi \log x)\, dx \\ &= \int_0^1 \sin(n\pi t) \sin(m\pi t)\, dt \\ &= \begin{cases} 1/2 & \text{if } n = m, \\ 0 & \text{if } n = m. \end{cases} \end{aligned}$$

- The eigenfunction expansion of $f \in L^2(1, e)$ is therefore

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} c_n x^{-1/2} \sin(n\pi \log x), \\ c_n &= 2 \int_1^e f(x) x^{-1/2} \sin(n\pi \log x)\, dx. \end{aligned}$$

4. Consider the singular Sturm-Liouville eigenvalue problem for Legendre's equation

$$\begin{aligned} -[(1-x^2)u']' &= \lambda u \qquad -1 < x < 1, \\ (1-x^2)u'(x) &\to 0 \quad \text{as } x \to \pm 1. \end{aligned}$$

(a) Solve the ODE for $\lambda = 0$ and show that both endpoints $x = \pm 1$ are in the limit circle case.

(b) For $n = 0, 1, 2, \ldots$, define the Legendre polynomials $P_n(x)$ by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right].$$

(Note that P_n *is* a polynomial of degree n.) Show that the Legendre polynomials are eigenfunctions of the Legendre equation with eigenvalues

$$\lambda_n = n(n+1).$$

HINT. Let $v(x) = (x^2-1)^n$ and differentiate the equation $(x^2 - 1)v' = 2nxv$ $n+1$ times.

(c) With v as in (b), show that

$$\int_{-1}^{1} \left[\frac{d^n v}{dx^n} \right]^2 = (2n)! \int_{-1}^{1} (1-x)^n (1+x)^n \, dx = (n!)^2 \int_{-1}^{1} (1+x)^{2n} \, dx$$

and deduce that

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}.$$

(d) Write out the orthogonality relations for the Legendre polynomials and the eigenfunction expansion of a function $f \in L^2(-1,1)$ with respect to the Legendre polynomials.

Solution

- (a) If $\lambda = 0$ then $[(1-x^2)u']' = 0$. Integrating this once, we get

  $$u' = \frac{c_2}{1-x^2}$$

  where c_2 is a constant. Integrating again, we get

  $$u(x) = c_1 + \frac{1}{2} c_2 \log \left| \frac{1+x}{1-x} \right|.$$

Thus, a fundamental pair of solutions of the ODE is

$$u_1(x) = 1, \qquad u_2(x) = \log\left|\frac{1+x}{1-x}\right|.$$

A function with a logarithmic singularity is square-integrable *e.g.*

$$\int_0^1 (\log x)^2\, dx < \infty,$$

so these functions are square-integrable on $(-1, 1)$. Both endpoints are therefore in the limit circle case.

- (b) If $v(x) = (x^2 - 1)^n$, then

$$(x^2 - 1)v' = (x^2 - 1) \cdot 2nx(x^2 - 1)^{n-1} = 2nxv. \qquad (1)$$

According to the Leibnitz rule,

$$\frac{d^{n+1}}{dx^{n+1}}(fg) = f\frac{d^{n+1}g}{dx^{n+1}} + (n+1)\frac{df}{dx}\frac{d^n g}{dx^n} + \frac{1}{2}n(n+1)\frac{d^2 f}{dx^2}\frac{d^{n-1}g}{dx^{n-1}} + \cdots + \frac{d^{n+1}f}{dx^{n+1}}g.$$

Since all derivatives of $(x^2 - 1)$ of order greater than or equal to three are zero, we have

$$\frac{d^{n+1}}{dx^{n+1}}\left[(x^2 - 1)v'\right] = (x^2 - 1)\frac{d^{n+2}v}{dx^{n+2}} + 2(n+1)x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n}.$$

Similarly, since all derivatives of x of order greater than or equal to two are zero, we have

$$\frac{d^{n+1}}{dx^{n+1}}(xv) = x\frac{d^{n+1}v}{dx^{n+1}} + (n+1)\frac{d^n v}{dx^n}.$$

Hence, differentiating (1) $n + 1$ times, we get

$$\begin{aligned} (x^2 - 1)\frac{d^{n+2}v}{dx^{n+2}} + 2(n+1)x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n} \\ = 2nx\frac{d^{n+1}v}{dx^{n+1}} + 2n(n+1)\frac{d^n v}{dx^n}. \end{aligned}$$

- Dividing this equation by $2^n n!$ and using the definition of $P_n(x)$, we find that

$$(x^2-1)P_n'' + 2(n+1)xP_n' + n(n+1)P_n = 2nxP_n' + 2n(n+1)P_n,$$

which simplifies to

$$-[(1-x^2)P_n']' = n(n+1)P_n.$$

This shows that $P_n(x)$ is an eigenfunction of the Legendre equation with eigenvalue $n(n+1)$.

- It follows, for example, from the Weierstrass approximation theorem that the Legendre polynomials $\{P_n : n = 0, 1, 2, \dots\}$ form a complete set in $L^2(-1, 1)$, so there are no other eigenvalues or eigenfunctions.

- (c) Integrating by parts n times, we get

$$\begin{aligned}
\int_{-1}^{1} \left(\frac{d^n v}{dx^n}\right)^2 &= \int_{-1}^{1} \frac{d^n}{dx^n}\left[(x^2-1)^n\right] \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right]\, dx \\
&= (-1)^n \int_{-1}^{1} (x^2-1)^n \cdot \frac{d^{2n}}{dx^{2n}}\left[(x^2-1)^n\right]\, dx \\
&= (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n\, dx.
\end{aligned}$$

The boundary terms drop out because v and its derivatives of order less than or equal to $n-1$ vanish at the endpoints $x = \pm 1$, and $(x^2-1)^n$ is a polynomial of degree $2n$ with leading term x^{2n}.

- Factoring $(x^2-1)^n$ and integrating by parts n times again, we get

$$\begin{aligned}
(-1)^n(2n)! \int_{-1}^{1} (x^2-1)^n\, dx &= (2n)! \int_{-1}^{1} (1-x)^n (1+x)^n\, dx \\
&= (2n)! \frac{n(n-1)\dots 1}{(n+1)(n+2)\dots 2n} \int_{-1}^{1} (1+x)^{2n}\, dx \\
&= (n!)^2 \left[\frac{(1+x)^{2n+1}}{2n+1}\right]_{-1}^{1} \\
&= \frac{2^{2n+1}(n!)^2}{2n+1}.
\end{aligned}$$

Hence

$$\begin{aligned}\int_{-1}^{1} P_n(x)^2\,dx &= \frac{1}{(2^n n!)^2}\int_{-1}^{1}\left(\frac{d^n v}{dx^n}\right)^2 dx \\ &= \frac{1}{2^{2n}(n!)^2}\frac{2^{2n+1}(n!)^2}{2n+1} \\ &= \frac{2}{2n+1}.\end{aligned}$$

- The orthogonality relations are

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 2/(2n+1) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

If $f \in L^2(-1,1)$, then

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx,$$

where the series converges with respect to the L^2-norm.

Remark. The Legendre polynomials also arise from the Gram-Schmidt orthogonalization of the powers $\{1, x, x^2, \dots, x^n, \dots\}$ in $L^2(-1,1)$. The expansion of a function f on some finite interval in terms of powers,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \dots,$$

is ill-posed (small changes in f may lead to large changes in the coefficients a_n). By contrast, the expansion of f in terms of orthogonal polynomials, such as the Legendre polynomials, is robust.