Problem Set 7

Math 207B, Winter 2012

Due: Fri., Mar. 2

1. (a) Explain why the total birth rate $B(t)$ of a population with constant reproductive rate λ per individual and exponential survival rate $e^{-\beta t}$ over time t satisfies the renewal equation

$$B(t) = N_0 \lambda e^{-\beta t} + \lambda \int_0^t e^{-\beta(t-s)} B(s)\, ds$$

where N_0 is the total initial population at $t = 0$. What are the dimensions of N_0, λ, β, and $B(t)$?

(b) Solve this integral equation and discuss the behavior of $B(t)$ as $t \to \infty$. Does your answer make sense?

Hint. One approach is to show that $B(t)$ satisfies the IVP

$$\dot{B} = (\lambda - \beta) B, \qquad B(0) = N_0 \lambda.$$

2. (a) Consider the following Fredholm equation of the second kind

$$u(x) - \lambda \int_0^1 xyu(y)\, dy = f(x), \qquad 0 \le x \le 1$$

where $\lambda \in \mathbb{C}$ is a constant and f is a given (continuous) function. If $\lambda \neq 3$, show that this equation has a unique (continuous) solution for $u(x)$ and find the solution. If $\lambda = 3$, determine for what functions f a solution exists and find the solutions in that case.

(b) For what functions f is the following Fredholm equation of the first kind

$$\int_0^1 xyu(y)\, dy = f(x), \qquad 0 \le x \le 1$$

solvable for $u(x)$? Describe the solutions in that case.

Hint. Note that these Fredholm equations are degenerate.

3. Show that the following IVP for a second-order scalar ODE for $u(t)$

$$\begin{aligned} \ddot{u}(t) &= f\left(t, u(t)\right), \\ u(0) &= u_0, \quad \dot{u}(0) = v_0 \end{aligned}$$

is equivalent to the Volterra integral equation

$$u(t) = \int_0^t (t-s) f\left(s, u(s)\right)\, ds + u_0 + v_0 t.$$

4. Consider the following BVP for $u(x)$ in $0 < x < 1$:

$$\begin{aligned} -u'' &= k^2\left[1 + \epsilon q(x)\right] u, \\ u(0) = 0, &\qquad u(1) = 0. \end{aligned} \tag{1}$$

Here $k > 0$ is a constant, ϵ is a small parameter, and $f(x)$, $q(x)$ are given (continuous) functions. Assume that $k \neq n\pi$ for any integer $n \in \mathbb{N}$, so that k^2 is not an eigenvalue of $-d^2/dx^2$ with Dirichlet BCs.

(a) Find the Green's function $G(x, \xi)$ for (1) with $\epsilon = 0$, which satisfies

$$\begin{aligned} -\frac{d^2 G}{d^2 x} &= k^2 G + \delta(x - \xi) \qquad \text{in } 0 < x < 1, \\ G(0, \xi) = 0, &\qquad G(1, \xi) = 0. \end{aligned}$$

(b) Use the Green's function from (a) to reformulate (1) as a Fredholm integral equation for $u(x)$ of the form

$$u(x) = \epsilon \int_0^1 K(x, \xi) u(\xi)\, d\xi + g(x).$$

(c) Write out the first few terms in the Neumann series (or Born approximation) for u as integrals involving $g(x)$, $q(x)$, and $G(x, \xi)$.