Solutions: Problem set 7

Math 207B, Winter 2012

1. (a) Explain why the total birth rate $B(t)$ of a population with constant reproductive rate λ per individual and exponential survival rate $e^{-\beta t}$ over time t satisfies the renewal equation

$$B(t) = N_0 \lambda e^{-\beta t} + \lambda \int_0^t e^{-\beta(t-s)} B(s)\, ds$$

where N_0 is the total initial population at $t = 0$. What are the dimensions of N_0, λ, β, and $B(t)$?

(b) Solve this integral equation and discuss the behavior of $B(t)$ as $t \to \infty$. Does your answer make sense?

Hint. One approach is to show that $B(t)$ satisfies the IVP

$$\dot{B} = (\lambda - \beta)B, \qquad B(0) = N_0\lambda.$$

Solution

- (a) After time t, there are $N_0 e^{-\beta t}$ survivors from the initial population, and they contribute $\lambda \cdot N_0 e^{-\beta t}$ to the birthrate. For $0 < s < t$, the population born between times s and $s + ds$ is $B(s)\, ds$, and of these $e^{\beta(t-s)} B(s)\, ds$ survive to time t, so they contribute $\lambda e^{-\beta(t-s)} B(s)\, ds$ to the birth rate. Integrating this over $0 < s < t$ and adding the contribution form the initial population, we get the total birth rate $B(t)$.

- Let P denote the dimension of population and T the dimension of time. Then
$$[N_0] = P, \quad [\lambda] = \frac{1}{T}, \quad [\beta] = \frac{1}{T}, \quad [B] = \frac{P}{T}.$$

- (b) Differentiating the integral equation with respect to t, we get
$$\dot{B}(t) = -N_0\lambda\beta e^{-\beta t} - \lambda\beta \int_0^t e^{-\beta(t-s)} B(s)\, ds + \lambda B(t).$$
Using the integral equation to eliminate the integral from this equation, we find that
$$\dot{B} = (\lambda - \beta)B.$$

Setting $t = 0$ in the integral equation, we get

$$B(0) = N_0\lambda.$$

The solution of this IVP is

$$B(t) = N_0\lambda e^{(\lambda-\beta)t}.$$

- If $\lambda > \beta$, meaning that the birth rate per individual exceeds the death rate, then the total birth rate (and the total population) grows exponentially in time. If $\lambda < \beta$, meaning that the death rate per individual exceeds the birth rate, then the total birth rate (and the total population) decays exponentially. If $\lambda = \beta$, then the total birth rate (and the total population) remains constant.

2. (a) Consider the following Fredholm equation of the second kind

$$u(x) - \lambda \int_0^1 xyu(y)\,dy = f(x), \qquad 0 \le x \le 1$$

where $\lambda \in \mathbb{C}$ is a constant and f is a given (continuous) function. If $\lambda \neq 3$, show that this equation has a unique (continuous) solution for $u(x)$ and find the solution. If $\lambda = 3$, determine for what functions f a solution exists and find the solutions in that case.

(b) For what functions f is the following Fredholm equation of the first kind

$$\int_0^1 xyu(y)\,dy = f(x), \qquad 0 \le x \le 1$$

solvable for $u(x)$? Describe the solutions in that case.

HINT. Note that these Fredholm equations are degenerate.

Solution

- (a) The integral equation implies that

$$u(x) = f(x) + cx \tag{1}$$

where the constant c is given by

$$c = \lambda \int_0^1 yu(y)\,dy.$$

Using the expression (1) for u in this equation for c, we get

$$\begin{aligned} c &= \lambda \int_0^1 yf(y)\,dy + \lambda \int_0^1 y \cdot cy\,dy \\ &= \lambda \int_0^1 yf(y)\,dy + \frac{1}{3}\lambda c, \end{aligned}$$

or

$$(3 - \lambda)\,c = 3\lambda \int_0^1 yf(y)\,dy.$$

- If $\lambda \neq 3$, then we get
$$c = \frac{3\lambda}{3-\lambda} \int_0^1 y f(y)\, dy$$
and (1) with this value of c is the unique solution of the integral equation for any continuous function f.

- If $\lambda = 3$, then the integral equation is solvable if and only if
$$\int_0^1 y f(y)\, dy = 0.$$
In that case, the solution is (1) where c is an arbitrary constant.

- Note that the integral operator
$$(Ku)(x) = \int_0^1 xyu(y)\, dy$$
has eigenvalue $\mu = 1/3$ and eigenfunction $u(x) = x$ with $Ku = \mu u$, so cx is an arbitrary solution of the homogeneous equation for $\lambda = 3$.

- (c) Since
$$\int_0^1 xyu(y)\, dy = cx, \qquad c = \int_0^1 yu(y)\, dy$$
the equation is only solvable if
$$f(x) = cx$$
for some constant c. In that case, one solution is the constant function
$$u(x) = 2c$$
The general solution is
$$u(x) = 2c + v(x)$$
where $v(x)$ is any function orthogonal to x, meaning that
$$\int_0^1 xv(x)\, dx = 0.$$

- Note that the range of K is one-dimensional, and the nullspace of K, consisting of eigenfunctions with eigenvalue $\mu = 0$, is infinite-dimensional.

3. Show that the following IVP for a second-order scalar ODE for $u(t)$

$$\ddot{u}(t) = f\left(t, u(t)\right),$$
$$u(0) = u_0, \quad \dot{u}(0) = v_0$$

is equivalent to the Volterra integral equation

$$u(t) = \int_0^t (t-s) f\left(s, u(s)\right)\, ds + u_0 + v_0 t.$$

Solution

- Integrating the ODE once, we get

$$\dot{u}(t) = v_0 + \int_0^t f\left(s, u(s)\right)\, ds.$$

Integrating again, we get

$$u(t) = u_0 + v_0 t + \int_0^t \left[\int_0^r f\left(s, u(s)\right)\, ds\right] dr.$$

Integrating by parts in the r-integral, we obtain

$$\begin{aligned}
\int_0^t \left[\int_0^r f\left(s, u(s)\right)\, ds\right] dr &= \int_0^t 1 \cdot \left[\int_0^r f\left(s, u(s)\right)\, ds\right] dr \\
&= \left[r \int_0^r f\left(s, u(s)\right)\, ds\right]_0^t - \int_0^t r f\left(r, u(r)\right)\, dr \\
&= t \int_0^t f\left(s, u(s)\right)\, ds - \int_0^t s f\left(s, u(s)\right)\, ds \\
&= \int_0^t (t-s) f\left(s, u(s)\right)\, ds,
\end{aligned}$$

which shows that a solution of the IVP satisfies the integral equation.

- Conversely, if $u(t)$ satisfies the integral equation, then setting $t = 0$ in the equation we get $u(0) = u_0$. Similarly, differentiating the integral equation once and setting $t = 0$, we get $\dot{u}(0) = v_0$. Differentiating once again, we get $\ddot{u} = f(t, u)$, so a solution of the integral equation satisfies the IVP.

- Note that the integral equation incorporates both the ODE and the initial conditions.

4. Consider the following BVP for $u(x)$ in $0 < x < 1$:

$$\begin{aligned} &-u'' = k^2\left[1 + \epsilon q(x)\right] u + f(x), \\ &u(0) = 0, \qquad u(1) = 0. \end{aligned} \tag{2}$$

Here $k > 0$ is a constant, ϵ is a small parameter, and $f(x)$, $q(x)$ are given (continuous) functions. Assume that $k \neq n\pi$ for any integer $n \in \mathbb{N}$, so that k^2 is not an eigenvalue of $-d^2/dx^2$ with Dirichlet BCs.

(a) Find the Green's function $G(x, \xi)$ for (2) with $\epsilon = 0$, which satisfies

$$\begin{aligned} &-\frac{d^2G}{d^2x} = k^2 G + \delta(x - \xi) \qquad \text{in } 0 < x < 1, \\ &G(0, \xi) = 0, \qquad G(1, \xi) = 0. \end{aligned} \tag{3}$$

(b) Use the Green's function from (a) to reformulate (2) as a Fredholm integral equation for $u(x)$ of the form

$$u(x) = \epsilon \int_0^1 K(x, \xi) u(\xi)\, d\xi + g(x). \tag{4}$$

(c) Write out the first few terms in the Neumann series (or Born approximation) for u as integrals involving $g(x)$, $q(x)$, and $G(x, \xi)$.

Solution

- (a) Solutions of the homogeneous equation

  $$-\frac{d^2u}{dx^2} - k^2 u = 0$$

  that vanish at $x = 0$ and $x = 1$ are

  $$u_1(x) = \sin kx, \qquad u_2(x) = \sin\left(k(1 - x)\right)$$

  respectively. The Wronskian of these solutions is

  $$\begin{aligned} u_1 u_2' - u_2 u_1' &= -k \sin kx \cos\left(k(1 - x)\right) - k \cos kx \sin\left(k(1 - x)\right) \\ &= -k \sin k, \end{aligned}$$

  which is nonzero if $k \neq n\pi$.

- The Green's function in (3), whose x-derivative jumps by -1 across $x = \xi$, is therefore

$$G(x,\xi) = \begin{cases} \sin kx \sin\left(k(1-\xi)\right)/k\sin k & \text{if } 0 \le x \le \xi, \\ \sin k\xi \sin\left(k(1-x)\right)/k\sin k & \text{if } \xi \le x \le 1. \end{cases}$$

or

$$G(x,\xi) = \frac{\sin\left(kx_<\right)\sin\left(k(1-x_>)\right)}{k\sin k}.$$

(b) We write (2) as

$$-u'' - k^2u = \epsilon q(x)u + f(x),$$
$$u(0) = 0, \qquad u(1) = 0.$$

It follows from the Green's function representation that

$$u(x) = \int_0^1 G(x,\xi)\left[\epsilon q(\xi)u(\xi) + f(\xi)\right]\,d\xi$$
$$= \epsilon\int_0^1 G(x,\xi)q(\xi)u(\xi)\,d\xi + \int_0^1 G(x,\xi)f(\xi)\,d\xi,$$

which gives (4) with

$$K(x,\xi) = G(x,\xi)q(\xi), \qquad g(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi.$$

- (c) The Neumann series expansion for

$$u = g + \epsilon Ku$$

is

$$u = g + \epsilon Kg + \epsilon^2K^2g + \epsilon^3K^3g + \dots$$

Explicitly, we have

$$Kg(x) = \int_0^1 G(x,\xi)q(\xi)g(\xi)\,d\xi,$$
$$K^2g(x) = \int_0^1\int_0^1 G(x,\xi_1)G(x,\xi_2)q(\xi_1)q(\xi_2)g(\xi_2)\,d\xi_1d\xi_2,$$
$$K^2g(x) = \int_0^1\int_0^1\int_0^1 G(x,\xi_1)G(x,\xi_2)G(x,\xi_3)q(\xi_1)q(\xi_2)q(\xi_3)g(\xi_3)\,d\xi_1d\xi_2d\xi_3.$$

- These terms correspond physically to single, double, and triple scattering corrections due to the perturbation $\epsilon q(x)$ in the coefficient.