Problem Set 8
Math 207B, Winter 2012
Due: Mon., Mar. 12

1. Let $G(x, \xi)$ be the Green's function for the Sturm-Liouville problem

$$-u'' = \lambda u, \qquad u(0) = u(1) = 0,$$

given by

$$G(x, \xi) = x_<(1 - x_>).$$

(a) What are the eigenvalues μ_n and eigenfunctions ϕ_n of G, where $n = 1, 2, \ldots$? (Find them from the corresponding eigenvalues and eigenfunctions of the Sturm-Liouville problem.)

(b) Compute

$$\int_0^1 \int_0^1 G(x, \xi)^2 \, dx d\xi.$$

(c) Use the identity

$$\int_0^1 \int_0^1 G(x, \xi)^2 \, dx d\xi = \sum_{n=1}^{\infty} \mu_n^2$$

to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Define the Abel integral operator K, acting on continuous functions $u(x)$ where $0 \le x \le 1$, by

$$(Ku)(x) = \int_0^x \frac{u(y)}{(x-y)^{1/2}}\, dy, \qquad 0 \le x \le 1.$$

(a) Is K a Hilbert-Schmidt operator?

(b) Show that $K^2 = \pi L$ where L is the integration operator

$$Lu(x) = \int_0^x u(y)\, dy.$$

HINT. The substitution $t = x\sin^2\theta + y\cos^2\theta$ shows that

$$\int_y^x \frac{dt}{(x-t)^{1/2}(t-y)^{1/2}} = \pi.$$

(c) Suppose that $f : [0,1] \to \mathbb{R}$ is a smooth function with $f(0) = 0$. Deduce that the solution of the Abel integral equation

$$\int_0^x \frac{u(y)}{(x-y)^{1/2}}\, dy = f(x), \qquad 0 \le x \le 1$$

is given by

$$u(x) = \frac{1}{\pi}\int_0^x \frac{f'(y)}{(x-y)^{1/2}}\, dy.$$

HINT. Solve the equation $Lu = (1/\pi)Kf$.