Solutions: Problem set 8
Math 207B, Winter 2012

1. Let $G(x,\xi)$ be the Green's function for the Sturm-Liouville problem

$$-u'' = \lambda u, \qquad u(0) = u(1) = 0,$$

given by

$$G(x,\xi) = x_<(1 - x_>).$$

(a) What are the eigenvalues μ_n and eigenfunctions ϕ_n of G, where $n = 1, 2, \ldots$? (Find them from the corresponding eigenvalues and eigenfunctions of the Sturm-Liouville problem.)
(b) Compute

$$\int_0^1 \int_0^1 G^2(x,\xi)\, dx d\xi.$$

(c) Use the identity

$$\int_0^1 \int_0^1 G^2(x,\xi)\, dx d\xi = \sum_{n=1}^{\infty} \mu_n^2 \tag{1}$$

to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{2}$$

Solution

- (a) The Sturm-Liouville problem has eigenvalues and orthonormal eigenfunctions

  $$\lambda_n = n^2\pi^2, \qquad \phi_n(x) = \sqrt{2}\sin n\pi x, \qquad n = 1, 2, 3, \ldots.$$

  Hence, the integral operator whose kernel is the Green's function has eigenvalues

  $$\mu_n = \frac{1}{n^2\pi^2}.$$

- (b) We compute that

$$
\begin{aligned}
\int_0^1 \int_0^1 G^2(x,\xi)\,dx d\xi &= \int_0^1 \left(\int_0^\xi G^2(x,\xi)\,dx \right) d\xi \\
&\qquad + \int_0^1 \left(\int_\xi^1 G^2(x,\xi)\,dx \right) d\xi \\
&= \int_0^1 \left(\int_0^\xi x^2(1-\xi)^2\,dx \right) d\xi \\
&\qquad + \int_0^1 \left(\int_\xi^1 \xi^2(1-x)^2\,dx \right) d\xi \\
&= \frac{1}{3} \int_0^1 \xi^3(1-\xi)^2\,d\xi + \frac{1}{3} \int_0^1 \xi^2(1-\xi)^3\,d\xi \\
&= \frac{1}{3} \int_0^1 \left(\xi^2 - 2\xi^3 + \xi^4 \right) d\xi \\
&= \frac{1}{90}.
\end{aligned}
$$

- (c) Using these results in (1), we get (2).

2. Define the Abel integral operator K, acting on continuous functions $u(x)$ where $0 \leq x \leq 1$, by

$$(Ku)(x) = \int_0^x \frac{u(y)}{(x-y)^{1/2}}\, dy, \qquad 0 \leq x \leq 1. \tag{3}$$

(a) Is K a Hilbert-Schmidt operator?

(b) Show that $K^2 = \pi L$ where L is the integration operator

$$Lu(x) = \int_0^x u(y)\, dy.$$

HINT. The substitution $t = x \sin^2\theta + y\cos^2\theta$ shows that

$$\int_y^x \frac{dt}{(x-t)^{1/2}(t-y)^{1/2}} = \pi.$$

(c) Suppose that $f : [0,1] \to \mathbb{R}$ is a smooth function with $f(0) = 0$. Deduce that the solution of the Abel integral equation

$$\int_0^x \frac{u(y)}{(x-y)^{1/2}}\, dy = f(x), \qquad 0 \leq x \leq 1$$

is given by

$$u(x) = \frac{1}{\pi}\int_0^x \frac{f'(y)}{(x-y)^{1/2}}\, dy. \tag{4}$$

HINT. Solve the equation $Lu = (1/\pi)Kf$.

Solution

- (a) The kernel of K is

$$k(x,y) = \begin{cases} (x-y)^{-1/2} & \text{if } 0 \leq y < x, \\ 0 & \text{if } x < y \leq 1. \end{cases}$$

It follows that

$$\int_0^1 k^2(x,y)\, dy = \int_0^x \frac{1}{x-y}\, dy$$

which is not finite, so

$$\int_0^1 \int_0^1 k^2(x,y)\, dx dy$$

is not finite, and K is not a Hilbert-Schmidt operator.

- (b) Writing K^2u as a double integral and exchanging the order of integration, we get

$$\begin{aligned}(K^2u)(x) &= \int_0^x \frac{(Ku)(t)}{(x-t)^{1/2}}\,dt \\ &= \int_0^x \frac{1}{(x-t)^{1/2}}\left(\int_0^t \frac{u(y)}{(t-y)^{1/2}}\,dy\right)dt \\ &= \int_0^x u(y)\left(\int_y^x \frac{1}{(x-t)^{1/2}(t-y)^{1/2}}\,dt\right)dy\end{aligned}$$

- Using the given substitution to change the integration variable from t to θ, we get

$$\int_y^x \frac{1}{(x-t)^{1/2}(t-y)^{1/2}}\,dt = \int_0^{\pi/2} 2\,d\theta = \pi.$$

Hence,

$$(K^2u)(x) = \pi\int_0^x u(y)\,dy$$

or $K^2 = \pi L$.

- (c) Note that by setting $x = 0$ in the integral equation, we see that we must have $f(0) = 0$ if the equation is solvable, and we assume this is the case.

- If $Ku = f$ then $K^2u = Kf$ or

$$\pi\int_0^x u(y)\,dy = Kf(x).$$

Differentiating this equation with respect to x, we get $\pi u = (Kf)'$ or

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(y)}{(x-y)^{1/2}}\,dy.$$

- We want to apply the x-derivative to the integral, but the integrand is too singular to allow this as it stands. We therefore first integrate by parts to regularize the integral:

$$\begin{aligned}\int_0^x \frac{f(y)}{(x-y)^{1/2}}\,dy &= \left[-2(x-y)^{1/2}f(y)\right]_0^x + 2\int_0^x (x-y)^{1/2}f'(y)\,dy \\ &= 2\int_0^x (x-y)^{1/2}f'(y)\,dy.\end{aligned}$$

Differentiating this equation, we get

$$\begin{aligned}\frac{d}{dx}\int_0^x \frac{f(y)}{(x-y)^{1/2}}\,dy &= 2\frac{d}{dx}\int_0^x (x-y)^{1/2} f'(y)\,dy \\ &= 2(x-x)^{1/2}f'(x) + 2\int_0^x \frac{d}{dx}\left[(x-y)^{1/2}\right] f'(y)\,dy \\ &= \int_0^x \frac{f'(y)}{(x-y)^{1/2}}\,dy.\end{aligned}$$

Hence, any solution of (3) is given by (4).

- Conversely, if u is given by (4), where $f(0) = 0$, then

$$\begin{aligned}Ku(x) &= \frac{1}{\pi}\int_0^x \frac{1}{(x-t)^{1/2}}\left(\int_0^t \frac{f'(y)}{(t-y)^{1/2}}\,dy\right)\,dt \\ &= \frac{1}{\pi}\int_0^x f'(y)\left(\int_y^x \frac{1}{(x-t)^{1/2}(t-y)^{1/2}}\,dt\right)\,dy \\ &= \int_0^x f'(y)\,dy \\ &= f(x)\end{aligned}$$

so (4) is the unique solution of (3).