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THE EQUATIONS FOR LARGE VIBRATIONS OF STRINGS

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1. Introduction. Many elementary books on partial differential equations ostensibly show that the wave equation in one spatial dimension describes the small transverse vibrations of an elastic string. Of these books I know of but one, namely [21], whose development of the wave equation does not invoke such unjustifed simplifications as the assumption that the motion of each particle of the string is confined to a plane perpendicular to the line joining the ends of the string. I fear that slipshod derivations of the equations of mathematical physics, like those relying on such ad hoc assumptions, succeed only in convincing novice scientists and mathematicians that applications are inherently dirty and incompatible with analysis. Indeed, good students of science can only be found among those who find such presentations incomprehensible.

This article has several objectives: (i) To show that the equations governing the large motion of a string of any material can be cleanly, simply, and honestly derived from fundamental principles. (ii) To show that the weak form of these equations can be obtained by a simple, yet rigorous, procedure that, unlike the standard methods, is not based upon tacit hypotheses that are invalid in the very instances when the weak form is most useful. (The weak form of the equations, which is formally equivalent to the physicists' Principle of Virtual Work, plays a central role in the modern theory of differential equations.) (iii) To examine some as yet unstudied aspects of the relationship between the nonlinear problem for elastic strings and its linearization about a straight equilibrium state. (iv) To discuss a number of analytical questions closely connected with the use of the weak formulation.

We begin our study in Section 2 by presenting a naive yet honest derivation of the classical equations of motion under the smoothness assumption that all derivatives appearing in the governing equations are continuous. In this setting we discuss the nature of elastic and viscoelastic strings and sketch the standard derivation of the weak equations. In Section 3 we carefully reconsider this derivation without these smoothness assumptions, which may well be unwarranted on both physical and mathematical grounds. Here we show the equivalence of the integral formulation of the governing equations as an impulse-momentum law to the weak formulation when the variables of the problem are merely required to be such that all the integrals appearing in the analysis are well defined in the sense of Lebesgue. Although some

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2. The Classical Equations. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis for the Euclidean 3-space $\mathbb{E}^3$. A *configuration* of a string, which we think of as a region of space that a string could occupy, is defined to be a curve (not necessarily simple) in $\mathbb{E}^3$. We define the *reference configuration* of the string to be the unit segment $\{x\mathbf{i}: x\in[0,1]\}$. We may think of the string in its unstretched state as occupying this configuration. We identify a *material particle* of the string by its coordinate x in this configuration. Let $\mathbf{r}(x,t)$ denote the position of particle x at time t. We take the domain of $\mathbf{r}$ to be $[0,1]\times[0,\infty)$. (The function $\mathbf{r}(\cdot,t)$ thus defines the *configuration at time t*.) Then $\mathbf{r}_x(x,t)\equiv(\partial\mathbf{r}/\partial x)(x,t)$ is tangent to the curve $\mathbf{r}(\cdot,t)$ at the point $\mathbf{r}(x,t)$. In this section we assume that all functions of (x,t) and of x that are exhibited here are continuous on the interiors of their domains. (Thus, by this convention, $\mathbf{r}_x$ is continuous on $(0,1)\times(0,\infty)$.)

We assume that the ends $x=0$ and $x=1$ of the string are held fixed at the points $\mathbf{0}$ and $L\mathbf{i}$. In the optimistic spirit that led us to assume that $\mathbf{r}$ is continuous on $(0,1)\times(0,\infty)$, we may further suppose that $\mathbf{r}(\cdot,t)$ is continuous on $[0,1]$. In this case these boundary conditions are defined by the following pointwise limits:

$$\lim_{x\to 0}\mathbf{r}(x,t)=\mathbf{0},\qquad \lim_{x\to 1}\mathbf{r}(x,t)=L\mathbf{i}\qquad (\text{for } t>0). \tag{2.1a}$$

Conditions (2.1a) are conventionally denoted by

$$\mathbf{r}(0,t)=\mathbf{0},\qquad \mathbf{r}(1,t)=L\mathbf{i}. \tag{2.1b}$$

We assume that at time $t=0$, the string is released from configuration $\mathbf{u}$ with velocity field $\mathbf{v}$. If $\mathbf{r}_t(x,\cdot)$ is assumed to be continuous on $[0,\infty)$, then these initial conditions have the pointwise interpretations

$$\lim_{t\to 0}\mathbf{r}(x,t)=\mathbf{u}(x),\qquad \lim_{t\to 0}\mathbf{r}_t(x,t)=\mathbf{v}(x), \tag{2.2a}$$

which are conventionally written as

$$\mathbf{r}(x,0)=\mathbf{u}(x),\qquad \mathbf{r}_t(x,0)=\mathbf{v}(x)\qquad \text{for } x\in(0,1). \tag{2.2b}$$

(In Section 3 we examine interpretations of (2.1b) and (2.2b) that are less restrictive than those of (2.1a) and (2.2a).) For $\mathbf{r}$ to have some chance of being smooth, we should require $\mathbf{u}$ to meet the *compatibility conditions*

$$\mathbf{u}(0)=\mathbf{0},\qquad \mathbf{u}(1)=L\mathbf{i}. \tag{2.3}$$

We complete our study of the geometry of the deformed string by noting that the *elongation* at (x,t), which is the local ratio at x of the length in the configuration at time t to that of the reference configuration, is $|\mathbf{r}_x(x,t)|$. We assume that

$$|\mathbf{r}_x(x,t)|>0 \tag{2.4}$$

to prevent this length ratio from being reduced to zero.

For $x\in(0,1)$ let $\mathbf{n}^+(x,t)$ be the contact force exerted on the material segment $[0,x)$ by the material segment $[x,1]$ at time t and let $-\mathbf{n}^-(x,t)$ be the contact force exerted on $(x,1]$ by $[0,x]$ at time t. (By definition of a contact force, $\mathbf{n}^+(x,t)$ is also the contact force exerted at time t on $[a,x)$ by $[x,b]$ for any a and b satisfying $0\leqslant a<x<b\leqslant 1$; i.e., $\mathbf{n}^+$ depends solely on (x,t).) Let $\mathbf{f}(x,t)$ be the force per unit reference length at (x,t) exerted by any other agent. $\mathbf{f}$ could depend on $\mathbf{r}$ in quite complicated ways; e.g., $\mathbf{f}$ could be given by $\mathbf{f}(x,t)=\mathbf{g}(\mathbf{r}(x,t),\mathbf{r}_t(x,t),x,t)$, where $\mathbf{g}$ is prescribed. (The dependence of $\mathbf{g}$ on $\mathbf{r}_t$ would account for air resistance, while its dependence on $\mathbf{r}$, which is of less physical importance, could account for the spring-like resistance of an ambient elastic medium or for variable gravitational attraction.) Let $\rho(x)$ be the mass density per unit length at x in the reference configuration.

The requirement that the resultant force on any material segment $(a,b)\subset(0,1)$ equal the time derivative of the linear momentum of that segment yields the *equations of motion* for the string:

$$\begin{aligned}\mathbf{n}^+(b,t)-\mathbf{n}^-(a,t)+\int_a^b\mathbf{f}(x,t)\,dx&=\frac{d}{dt}\int_a^b\rho(x)\mathbf{r}_t(x,t)\,dx\\&=\int_a^b\rho(x)\mathbf{r}_{tt}(x,t)\,dx\quad\text{for all }(a,b)\subset(0,1).\end{aligned}\tag{2.5}$$

Now the continuity of $\mathbf{n}^+$ implies that $\mathbf{n}^+(a,t)=\lim_{b\to a}\mathbf{n}^+(b,t)$. Since $\mathbf{f}$ and $\mathbf{r}_{tt}$ have been assumed to be continuous, we may let $b\to a$ in (2.5) to obtain

$$\mathbf{n}^+(a,t)=\mathbf{n}^-(a,t)\quad\text{for all }a\in(0,1). \tag{2.6}$$

We may accordingly drop the superscripts from $\mathbf{n}$. If we now differentiate (2.5) with respect to b and then replace b by x, we obtain the *classical equations of motion of a string*:

$$\mathbf{n}_x(x,t)+\mathbf{f}(x,t)=\rho(x)\mathbf{r}_{tt}(x,t),\quad\text{for }x\in(0,1),\ t>0. \tag{2.7}$$

We describe the material properties of the string by specifying how the force $\mathbf{n}$ is related to the motion $\mathbf{r}$. Such a relation is necessary if we are to obtain a formally determinate system containing (2.7). The defining property of a string, which distinguishes it from a rod (which resists bending), is that $\mathbf{n}(x,t)$ must be tangent to the curve $\mathbf{r}(\cdot,t)$ at x. (The motivation for this condition comes from the classical equations expressing the equality of the resultant torque on any segment to the time derivative of angular momentum for that segment. These equations are $\mathbf{m}_x+\mathbf{r}_x\times\mathbf{n}+\mathbf{g}=\mathbf{w}_t$ where $\mathbf{m}(x,t)$ is the couple exerted on $[0,x)$ by $[x,1]$ at t, $\mathbf{g}(x,t)$ is the applied couple per unit length, and $\mathbf{w}$ is the angular momentum. Suppose $\mathbf{g}=\mathbf{0}$. If we assume that our one-dimensional body has zero thickness, then we can take $\mathbf{m}=\mathbf{0}$ and $\mathbf{w}=\mathbf{0}$. In this case, the angular momentum equation reduces to $\mathbf{r}_x\times\mathbf{n}=\mathbf{0}$, which is our assumption for strings.) We accordingly take

$$\mathbf{n}(x,t)=n(x,t)\mathbf{r}_x(x,t)/|\mathbf{r}_x(x,t)|. \tag{2.8}$$

$n(x,t)$ is the *tension* at (x,t). Let

$$\mathbf{r}^t(x,s)=\mathbf{r}(x,t-s)\quad\text{for }s\geqslant 0. \tag{2.9}$$

$\mathbf{r}^t(x,\cdot)$ is called the *history* of $\mathbf{r}(x,\cdot)$ up to time t. We account for a very large class of materials for strings by assuming that there is a functional N such that

$$n(x,t)=N(|\mathbf{r}_x^t(x,\cdot)|,x), \tag{2.10}$$

i.e., that the tension at (x,t) is determined by the past history of the elongation $|\mathbf{r}_x|$ at x. We do not allow N to depend upon $\mathbf{r}^t(x,\cdot)$ or upon $\mathbf{r}_x^t(x,\cdot)$ because such a dependence would imply that the material properties of the string could be altered by causing it to undergo a rigid motion. We do not allow N to depend upon t because such a dependence would imply that material properties would be influenced by the choice of a clock. N could depend upon $|\mathbf{r}_x^t(\cdot,\cdot)|$, but such generality has not proved particularly fruitful in mechanics. In the special case for which there is a function N_0 such that

$$n(x,t)=N_0(|\mathbf{r}_x(x,t)|;x), \tag{2.11}$$

the string is called *elastic*. That an increase in tension accompany an increase in elongation is ensured by requiring that $N_0(\cdot,x)$ be increasing. If there is a function N_1 such that

$$n(x,t)=N_1(|\mathbf{r}_x(x,t)|,|\mathbf{r}_x(x,t)|_t,x), \tag{2.12}$$

the string is called *viscoelastic* (of a *differential type*). That an increase in tension accompany an increase in the rate of elongation is ensured by demanding that $\beta\mapsto N_1(\alpha,\beta,x)$ be increasing. If we substitute (2.11) or (2.12) into (2.8) and then substitute this into (2.7) we obtain a quasilinear system of partial differential equations. If we use the more general constitutive assumption (2.10), then we get what may be called a quasilinear partial functional-differential equation. In this case the initial conditions (2.2) would have to be supplemented by giving $\mathbf{r}^0(\cdot,\cdot)$.

Let $\boldsymbol{\eta}$ have compact support in $(0,1)\times(0,\infty)$. We can formally obtain the *weak form* of (2.7) or the *Principle of Virtual Work* by dotting (2.7) with $\boldsymbol{\eta}$ and integrating the resulting expression by parts over $(0,1)\times(0,\infty)$. We obtain

$$\int_0^\infty\int_0^1[\mathbf{n}\cdot\boldsymbol{\eta}_x-\mathbf{f}\cdot\boldsymbol{\eta}+\rho\mathbf{r}_t\cdot\boldsymbol{\eta}_t]\,dx\,dt=0 \tag{2.13}$$

for all such $\boldsymbol{\eta}$. By using the arbitrariness of $\boldsymbol{\eta}$ and the smoothness of our other variables, we could easily reverse our steps and recover (2.7) and (2.5) from (2.13).

3. The Weak Form of the Equations. It has long been known that the solutions of the equations for purely longitudinal motion of elastic strings can exhibit shocks, i.e., discontinuities in $\mathbf{r}_x$ and $\mathbf{r}_t$ (cf. [**14**], [**16**]). The same is also true for strings governed by certain forms of (2.10), in which N depends on the past history of $\mathbf{r}_x$ (cf. [**9**], e.g.). The shock structure of spatial motions of elastic strings has recently been analyzed by [**13**]. Thus the smoothness assumptions made in Section 2 are completely unwarranted for elastic strings at least. It is clear that the integral formulations (2.5) and (2.13) would make sense under far weaker smoothness assumptions than used in Section 2; it is not clear, however, that (2.5) and (2.13) are equivalent. In this section, we study the formulation of the problem under these weaker smoothness assumptions and we give a simple direct proof of the equivalence of precisely formulated generalizations of (2.5) and (2.13). This proof replaces the demonstration of formal equivalence given in Section 2, which is universally propounded by mathematicians and physicists alike despite the pivotal role played in it by the classical equation (2.7), an equation that may be devoid of mathematical meaning. In consonance with the goals of this section and in contrast with the methods of Section 2, we must state regularity restrictions on our variables with great care.

If we formally integrate (2.5) with respect to t over $[0,\tau]$ and take account of (2.2), then we obtain the *(Linear) Impulse-Momentum Law*:

$$\int_0^\tau[\mathbf{n}^+(b,t)-\mathbf{n}^-(a,t)]\,dt+\int_0^\tau\int_a^b\mathbf{f}(x,t)\,dx\,dt \\ =\int_a^b\rho(x)[\mathbf{r}_t(x,\tau)-\mathbf{v}(x)]\,dx. \tag{3.1}$$

The left side of (3.1) is the *(linear) impulse of the force system* $\{\mathbf{n}^\pm,\mathbf{f}\}$ and the right side of (3.1) is the *change in linear momentum* for the segment (a,b) over time interval $[0,\tau)$. We regard (3.1) as the natural generalization of the equations of motion (2.5). We now state virtually the weakest possible restrictions on the functions entering (3.1) for the integrals of (3.1) to make sense as Lebesgue integrals and for our boundary and initial conditions to have consistent generalizations. We then study these generalizations by appealing to results from real analysis. We resume the main thread of our development in the paragraph containing (3.5). The reader unfamiliar with real analysis may wish to skim over or even skip the intervening material, which appears in small type.

We assume that there are numbers ρ^- and ρ^+ such that

$$0<\rho^- \leq \rho(x) \leq \rho^+ < \infty \quad \text{for all } x\in[0,1]. \tag{3.2}$$

We assume that $\mathbf{r}_x$ and $\mathbf{r}_t$ are locally integrable on $[0,1]\times[0,\infty)$, that $\mathbf{r}$ satisfies the boundary conditions (2.1) in the sense of trace (cf. [1], [17]), i.e., that

$$\lim_{x\to 0}\int_{t_1}^{t_2}\mathbf{r}(x,t)\,dt=\mathbf{0}, \qquad \lim_{x\to 1}\int_{t_1}^{t_2}[\mathbf{r}(x,t)-L\mathbf{i}]\,dt=\mathbf{0} \qquad \text{for each } (t_1,t_2)\in[0,\infty), \tag{3.3}$$

that $\mathbf{u}$ is integrable on $[0,1]$, that the first initial condition of (2.2) is assumed in the sense of trace:

$$\lim_{t\to 0}\int_a^b[\mathbf{r}(x,t)-\mathbf{u}(x)]\,dx=\mathbf{0} \quad \text{for all } (a,b)\subset(0,1), \tag{3.4}$$

and that $\mathbf{v}$ is integrable on $[0,1]$. (Conditions (3.2) and (3.3) are consistent with the local integrability of $\mathbf{r}_x$ and $\mathbf{r}_t$; cf. [1], [17].) We do not prescribe a corresponding generalization of the second initial condition of (2.2) because we shall show that it is inherent in (3.1) as the presence there of $\mathbf{v}$ might suggest.

We assume that $\mathbf{n}^+$, $\mathbf{n}^-$, and $\mathbf{f}$ are locally integrable on $[0,1]\times[0,\infty)$. After we determine in what sense $\mathbf{n}^+$ equals $\mathbf{n}^-$ we shall see that this requirement imposes growth restrictions on the functional N of (2.10), which we do not explore.

We must show that the first and third integrals of (3.1) make sense. This will enable us to obtain a weaker version of the intepretation of $\mathbf{n}^\pm$ than that prevailing in Section 2. Since $\mathbf{n}^+$ is locally integrable on $[0,1]\times[0,\infty)$, Fubini's Theorem implies that for each $\tau\in(0,\infty)$, there is a set $A^+(\tau)\in[0,1]$ with Lebesgue measure $|A^+(\tau)|=1$ such that $\mathbf{n}^+(x,\cdot)$ is integrable over $[0,\tau]$ for $x\in A^+(\tau)$. Moreover, the Lebesgue Differentiation Theorem implies that there is a subset $A_0^+(\tau)$ of $A^+(\tau)$ with $|A_0^+(\tau)|=1$ such that for $x\in A_0^+(\tau)$, $\int_0^\tau \mathbf{n}^+(x,t)\,dt$ has the "right" value in the sense that it is the limit of its averages over intervals centered at x. The corresponding statements obtained by replacing the superscript "+" by "−" are likewise true. Let $A(\tau)=A_0^+(\tau)\cap A_0^-(\tau)$ ($|A(\tau)|=1$). Let B be the set of t's for which $\rho(\cdot)\mathbf{r}_t(\cdot,t)$ is integrable over $[0,1]$ and for which $\int_0^1\rho(x)\mathbf{r}_t(x,t)\,dx$ has the "right" value. (By Fubini's Theorem and Lebesgue's Differentiation Theorem we have that $|B\cap[0,T]|=T$ for all $T>0$.) Thus each term in (3.1) is well defined for each $\tau\in B$ and for each a and b in $A(\tau)$ with $a<b$.

We now derive some important consequences from (3.1). Since Fubini's Theorem allows us to interchange the order of integration of the double integral in (3.1), we find that $b\mapsto\int_0^\tau \mathbf{n}^+(b,t)\,dt$ is absolutely continuous on $A(\tau)$. Consequently,

$$\int_0^\tau \mathbf{n}^+(a,t)\,dt=\lim\int_0^\tau \mathbf{n}^+(b,t)\,dt \quad \text{as } b\to a \text{ through } A(\tau).$$

Then (3.1) implies that

$$\int_0^\tau \mathbf{n}^+(a,t)\,dt=\int_0^\tau \mathbf{n}^-(a,t)\,dt \quad \text{for each } \tau\in B \text{ and for each } a\in A \tag{3.5}$$

Thus the superscripts "+" and "−" are superfluous in (3.1) and will accordingly be dropped. Next, we observe that the properties of the Lebesgue integral imply that if $a,b\in A(\tau)$, then $a,b\in A(\sigma)$ for each $\sigma\in[0,\tau]$. Let us fix $\tau>0$ and replace τ in (3.1) by σ. Let $a,b\in A(\tau)$. By our preceding remarks (3.1) makes sense for each $\sigma\in B$. Letting $\sigma\to 0$ through B we obtain

$$\lim_{B\ni\sigma\to 0}\int_a^b\rho(x)[\mathbf{r}_t(x,\sigma)-\mathbf{v}(x)]\,dx=\mathbf{0} \quad \text{for all } a,b \text{ in } A(\tau). \tag{3.6}$$

This generalization of the second initial condition of (2.2) is thus implicit in (3.1). It is somewhat weaker than the analogous (3.4), which may be interpreted as saying that $\mathbf{r}(\cdot,t)$ converges weakly to $\mathbf{u}$ in $L_1(0,1)$. (Cf. [22] for a discussion of modes of convergence weaker than weak convergence.)

Without indulging in the artificial exploitation of (2.7), we now show how (3.1) is equivalent to a precisely formulated version of (2.13). Let ϕ be a piecewise linear function with support in

(a,c) and let ψ be a piecewise linear function with support in $[0,\tau)$. Let $\mathbf{e}$ be a fixed unit vector. Then (3.1) implies that

$$\int_0^\tau \int_a^c \phi_b(b)\psi_t(t)\left\{\int_0^t \mathbf{e}\cdot[\mathbf{n}(b,s)-\mathbf{n}(a,s)]\,ds+\int_0^t\int_a^b \mathbf{e}\cdot\mathbf{f}(x,s)\,dx\,ds\right\}db\,dt$$
$$=\int_0^\tau \int_a^c \phi_b(b)\psi_t(t)\int_a^b \rho(x)\mathbf{e}\cdot[\mathbf{r}_t(x,t)-\mathbf{v}(x)]\,dx\,db\,dt. \tag{3.7}$$

Since ψ is absolutely continuous we can integrate the left side of (3.7) by parts with respect to t over $[0,\tau]$. Since $\psi(\tau)=0$ and $\phi(a)=0$, this integration yields the result that the triple integral on the left side of (3.7) equals

$$-\int_0^\tau \int_a^c \phi_b(b)\psi(t)\mathbf{e}\cdot\mathbf{n}(b,t)\,db\,dt. \tag{3.8a}$$

Similarly, the quadruple integral on the left side of (3.7) reduces to

$$\int_0^\tau \int_a^c \phi(b)\psi(t)\mathbf{e}\cdot\mathbf{f}(b,t)\,db\,dt \tag{3.8b}$$

and the right side of (3.7) equals

$$-\int_0^\tau \int_a^c \phi(b)\psi_t(t)\rho(b)\mathbf{e}\cdot[\mathbf{r}_t(x,t)-\mathbf{v}(x)]\,db\,dt. \tag{3.9}$$

Let us set

$$\boldsymbol{\eta}(x,t)=\phi(x)\psi(t)\mathbf{e}. \tag{3.10}$$

Since $\boldsymbol{\eta}$ has support in $(a,c)\times[0,\tau)$, we can use (3.8) and (3.9) to write (3.7) as

$$\int_0^\infty \int_0^1 [\mathbf{n}(x,t)\cdot\boldsymbol{\eta}_x(x,t)-\mathbf{f}(x,t)\cdot\boldsymbol{\eta}(x,t)]\,dx\,dt$$
$$=\int_0^\infty \int_0^1 \rho(x)[\mathbf{r}_t(x,t)-\mathbf{v}(x)]\cdot\boldsymbol{\eta}_t(x,t)\,dx\,dt \tag{3.11}$$

for all $\boldsymbol{\eta}$'s of the form (3.10) and more generally for all $\boldsymbol{\eta}$'s in the space V that is the completion of finite linear combinations of functions of the form (3.10) in the norm $\|\boldsymbol{\eta}\|=\operatorname{ess\,sup}(|\boldsymbol{\eta}_x|+|\boldsymbol{\eta}_t|)$. (Some properties of this space V are discussed in [3].) Note that this class of $\boldsymbol{\eta}$'s is larger than that used in (2.13) because these $\boldsymbol{\eta}$'s need not have support in $(0,1)\times(0,\infty)$. Consequently, the form of (3.11) is more general than that of (2.13). Equation (3.11) is the *Principle of Virtual Work* or the *Weak Form* of (2.7). The Weak Form of the Initial-Boundary Value Problem for Elastic Strings is obtained by inserting (2.8) and (2.11) into (3.11). Conversely, we can recover (3.1) (without the superscripts "+" and "−") from (3.11) by taking $\boldsymbol{\eta}$ to have the form (3.10) with

$$\phi(x)=\begin{cases} 0 & \text{for } 0\leqslant x\leqslant a,\\ \dfrac{x-a}{\varepsilon} & \text{for } a\leqslant x\leqslant a+\varepsilon,\\ 1 & \text{for } a+\varepsilon\leqslant x\leqslant b-\varepsilon,\\ \dfrac{b-x}{\varepsilon} & \text{for } b-\varepsilon\leqslant x\leqslant b,\\ 0 & \text{for } b\leqslant x,\end{cases} \tag{3.12a}$$

$$\psi(t)=\begin{cases} 1 & \text{for } 0\leqslant t\leqslant \tau,\\ 1+\dfrac{\tau-t}{\varepsilon} & \text{for } \tau\leqslant t\leqslant \tau+\varepsilon,\\ 0 & \text{for } \tau+\varepsilon\leqslant t,\end{cases} \tag{3.12b}$$

and then letting $\varepsilon \to 0$. In this process, we must evaluate

$$\lim_{\varepsilon \to 0} \int_0^{\tau+\varepsilon} \frac{1}{\varepsilon} \int_a^{a+\varepsilon} \mathbf{n}(x,t) \cdot \mathbf{e} \psi(t)\, dx\, dt, \tag{3.13}$$

which Fubini's Theorem and (3.12b) allow us to rewrite as

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_a^{a+\varepsilon} \int_0^{\tau} \mathbf{n}(x,t) \cdot \mathbf{e}\, dt\, dx + \lim_{\varepsilon \to 0} \varepsilon \left[\frac{1}{\varepsilon^2} \int_a^{a+\varepsilon} \int_\tau^{\tau+\varepsilon} \mathbf{n}(x,t) \cdot \mathbf{e} \psi(t)\, dt\, dx \right]. \tag{3.14}$$

Now the Lebesgue Differentiation Theorem implies that the first limit in (3.14) is

$$\int_0^{\tau} \mathbf{n}(a,t) \cdot \mathbf{e}\, dt \tag{3.15}$$

for almost all a in $(0,1)$ and that the least upper bound of the absolute value of the bracketed term in the second limit of (3.14) is finite for almost all $(a,\tau) \in (0,1) \times (0,\infty)$. Thus (3.13) equals (3.14). The other terms are treated similarly. The arbitrariness of $\mathbf{e}$ allows it to be canceled in the final expression.

If $\mathbf{f}$ is not locally integrable, then the development culminating in (3.5) and (3.6) is not valid. $\mathbf{f}$ would not be locally integrable if there were concentrated or impulsive forces applied to the string. Such forces could be described by measures in the context of (3.1) and by distributions in the context of (3.11). It is not evident what class of force measures, when introduced into (3.1), will both yield an effective generalization of (3.5) and support a proof of equivalence of suitably generalized versions of (3.1) and (3.11). A mathematically natural scale of generalizations of (3.11) can be obtained by the following standard procedure: We write (3.11) as

$$\langle \boldsymbol{\sigma}, \boldsymbol{\eta} \rangle = 0 \tag{3.16}$$

where $\boldsymbol{\sigma}$ accounts for contributions from $\mathbf{n}$, $\mathbf{f}$, and $\rho(\mathbf{r}_t - \mathbf{v})$. If we restrict $\boldsymbol{\eta}$ to belong to a class of functions E whose members are smoother than those of V, then we could correspondingly extend $\langle \cdot, \boldsymbol{\eta} \rangle$ for $\boldsymbol{\eta} \in E$ to a class of $\boldsymbol{\sigma}$'s larger than those studied above. (For example, if E were to consist of infinitely differentiable functions $\boldsymbol{\eta}$ with support in $(0,1) \times (0,\infty)$, then the class of σ's are *distributions*. This choice of E would, however, strip (3.11) of its possession of a generalization of (3.6).) This process of extending the bilinear form $\langle \cdot, \cdot \rangle$, however elegant, still avoids confronting the fundamental physical question of generalizing (3.5).

Even if a physically satisfactory extension of (3.1) were available to account for data as measures, the exceptionally challenging problem of actually analyzing such an initial-boundary value problem remains. Some, but not all, of the difficulties to be faced in such an analysis can be observed in the recent study [4] of a boundary value problem for a single semi-linear elliptic equation. In the beginning of the next section, we actually work out the solution of a degenerately simple static problem for a concentrated force on a string. One question of physical importance that arises in such analyses is to determine how a sequence of solutions corresponding to a sequence of locally integrable data converges as the sequence of data converges in some suitable topology to a measure (such as the Dirac measure). The study of concentrated and impulsive loads in the linear equations for strings, and in fact for linear differential equations in general, is of central importance because the theory yields by superposition solutions for much larger classes of data. This is not the case for nonlinear problems. One manifestation of this in the context of elastic strings, say, is that not all the derivatives on $\mathbf{r}$ can be shifted from $\boldsymbol{\sigma}$ to the test function $\boldsymbol{\eta}$ in (3.16) by integration by parts. Thus, in nonlinear problems concentrated and impulsive forces must be studied only for their intrinsic physical or mathematical interest.

4. The Linearized Equations. Suppose that (2.11) holds and that $\alpha \mapsto N_0(\alpha, x)$ is strictly increasing from $-\infty$ to ∞ as α increases from 0 to ∞. Let $p:[0,1] \to \mathbb{R}$ be integrable. Let $\mathbf{u}(x) = \sigma(x)\mathbf{i}$ where σ satisfies the equations

$$[N_0(\sigma_x(x), x)]_x + p(x) = 0, \quad \sigma(0) = 0, \quad \sigma(1) = L. \tag{4.1a,b,c}$$

If $\mathbf{v} = \mathbf{0}$ and if $\mathbf{f}(x,t) = p(x)\mathbf{i}$, then to each solution σ of (4.1) there corresponds a static solution of an appropriately generalized version of the initial boundary value problem (2.1), (2.2), (2.7), (2.8), (2.11) given by $\mathbf{r}(x,t)=\sigma(x)\mathbf{i}$. In other words, this $\mathbf{r}$ satisfies (3.1) or (3.11) subject to the side conditions (3.3) and (3.4), as is easily verified. (This solution $\mathbf{r}$ can be shown to be unique among sufficiently regular solutions of the initial-boundary value problem.)

The boundary value problem (4.1) has a unique solution σ as the following elementary argument shows. Equation (4.1a) is equivalent to

$$N_0(\sigma_x(x),x)+P(x)=\nu\,(\text{const.}),\ P(x)\equiv\int_0^x p(y)\,dy, \tag{4.2}$$

which in turn is equivalent to

$$\sigma_x(x)=R(\nu-P(x),x). \tag{4.3}$$

Here $R(\cdot,x)$ is the inverse of $N_0(\cdot,x)$; R exists by virtue of the assumptions imposed on N_0. Moreover, $\beta\mapsto R(\beta,x)$ is strictly increasing from 0 to ∞ as β increases from $-\infty$ to ∞. If N_0 is continuously differentiable, then so is R by the implicit function theorem. The integration of (4.3) shows that

$$\sigma(x)=\int_0^x R(\nu-P(y),y)\,dy \tag{4.4}$$

satisfies (4.1a, b) and would satisfy (4.1c) if ν could be chosen so that

$$\int_0^1 R(\nu-P(y),y)\,dy=L. \tag{4.5}$$

But $\nu\mapsto\int_0^1 R(\nu-P(y),y)\,dy$, just like $R(\cdot,x)$, is strictly increasing from 0 to ∞ as ν increases from $-\infty$ to ∞, so that (4.5) has a unique solution for ν in terms of L. This means that (4.1) has a unique solution for σ in terms of L and P, which is obtained by substituting the solution ν of (4.5) into (4.4). Since P is absolutely continuous, equation (4.9) shows that σ is continuously differentiable, if R is continuous. If p were continuous and R were continuously differentiable, then σ would be twice continuously differentiable. Note that if the string is uniform (so that N_0 is independent of x) and if $P=0$, then $\sigma=xL$. Since $N_0(\alpha,x)\to-\infty$ as $\alpha\to 0$, equation (4.3) ensures that $\sigma_x(x)>0$ for all x in $[0,1]$ (see (2.4)), so that σ is invertible. If $N_0(1,x)=0$, which reflects the eminently reasonable assumption that the reference state is free of tension, and if $P=0$, then the solution ν of (4.5) has the same sign as $L-1$. (This elementary analysis of the existence and uniqueness of σ is a primitive prototype of the application of methods of monotone operator theory to quasilinear elliptic equations.)

Let us note that (4.2) is equivalent to (2.11) and (3.1) without the superscripts "+" and "−". Equation (4.2) makes sense even when P is not absolutely continuous (i.e., when P is not the indefinite integral of an integrable function). The analysis goes through without modification as long as P is a real-valued function, e.g., if P were the Heaviside function H_c, $c\in(0,1)$. ($H_c(x)=0$ for $x<c$ and $=1$ for $x>c$.) In this case p would be the Dirac delta concentrated at c. The simplicity of this highly degenerate problem is misleading. See the discussion at the end of the last section.

Now let us consider the problem in which $\mathbf{u},\mathbf{v}$, and $\mathbf{f}$ have the form

$$\mathbf{u}(x)=\sigma(x)\mathbf{i}+\varepsilon\mathbf{u}_1(x),\mathbf{v}(x)=\varepsilon\mathbf{v}_1(x),\mathbf{f}(x,t)=p(x)\mathbf{i}+\varepsilon\mathbf{f}_1(x,t). \tag{4.6}$$

Here ε represents a small real parameter. Suppose that $N_0(\cdot,x)$ is $(p+1)$ times continuously differentiable. We seek formal solutions of the initial-boundary value problem whose dependence on the parameter ε is specified by

$$\mathbf{r}(x,t,\varepsilon)=\sigma(x)\mathbf{i}+\sum_{k=1}^{p}\frac{\varepsilon^k}{k!}\mathbf{r}_k(x,t)+o(\varepsilon^{p+1}). \tag{4.7}$$

Since

$$\mathbf{r}_k(x,t)=\frac{\partial^k\mathbf{r}(x,t,\varepsilon)}{\partial\varepsilon^k}\Big|_{\varepsilon=0},$$

we can find the problem formally satisfied by $\mathbf{r}_k$ by substituting (4.7) into the equations of the nonlinear problem, differentiating the resulting equations k times with respect to ε, and then setting $\varepsilon=0$. We find that the equation for $\mathbf{r}_k$ involves $\mathbf{r}_1,\ldots,\mathbf{r}_{k-1}$; thus the system of equations for $\mathbf{r}_1,\ldots,\mathbf{r}_p$ can be solved recursively. In particular, the equation for $\mathbf{r}_1$ reduces to

$$\left\{N_0'(\sigma_x(x),x)(\mathbf{r}_{1x}\cdot\mathbf{i})\mathbf{i}+N_0(\sigma_x(x),x)[\sigma_x(x)]^{-1}[(\mathbf{r}_{1x}\cdot\mathbf{j})\mathbf{j}+(\mathbf{r}_{1x}\cdot\mathbf{k})\mathbf{k}]\right\}_x -\rho(x)\mathbf{r}_{1tt}=-\mathbf{f}_1(x,t), \qquad (4.8)$$

where N_0' is the partial derivative of N_0 with respect to its first argument. $\mathbf{r}_1$ must satisfy the boundary conditions

$$\mathbf{r}_1(0,t)=\mathbf{0}, \qquad \mathbf{r}_1(1,t)=\mathbf{0} \qquad (4.9)$$

and initial conditions

$$\mathbf{r}_1(x,0)=\mathbf{u}_1(x), \qquad \mathbf{r}_{1t}(x,0)=\mathbf{v}_1(x). \qquad (4.10)$$

If $\mathbf{f}_1\cdot\mathbf{i}=0$, then (4.8) implies that $\mathbf{r}_1\cdot\mathbf{i}$ satisfies the scalar wave equation

$$[N_0'(\sigma_x(x),x)w_x]_x=\rho(x)w_{tt} \quad \text{for } x\in(0,1), t>0 \qquad (4.11)$$

where σ_x is given by (4.3). If we set

$$s=\sigma(x),\ \hat{\mathbf{r}}_1(s,t)=\mathbf{r}_1(\sigma^{-1}(s),t),\ \hat{\rho}(s)=\frac{\rho(\sigma^{-1}(s))}{\sigma_x(\sigma^{-1}(s))}, \qquad (4.12)$$

if $\mathbf{f}_1\cdot\mathbf{j}=0$ and $\mathbf{f}_1\cdot\mathbf{k}=0$, and if we use (4.2), then we obtain from (4.8) that $\hat{\mathbf{r}}_1\cdot\mathbf{j}$ and $\hat{\mathbf{r}}_1\cdot\mathbf{k}$ each satisfy the scalar wave equation

$$\left\{\left[\nu-P(\sigma^{-1}(s))\right]w_s\right\}_s=\hat{\rho}(s)w_{tt} \quad \text{for } s\in(0,L), t>0. \qquad (4.13)$$

Note that equations (4.11) and (4.13) are uncoupled.

Equation (4.11) describes the small longitudinal motion of the string (or of a rod) about its straight, stretched equilibrium state. Note that both the nonuniformity of the string and the presence of P cause the coefficients of (4.11) to depend upon x. (Equation (4.11) is frequently cited as a source of Sturm-Liouville problems for ordinary differential equations; these are obtained from (4.11) by separation of variables.) Since N_0' is positive by hypothesis, equation (4.11) is hyperbolic.

Equation (4.13) describes the small transverse vibrations of the string. $\nu-P(\sigma^{-1}(s))$ is the tension of the string at $\sigma^{-1}(s)$ in the configuration $\sigma\mathbf{i}$ and $\hat{\rho}$ is the mass per unit length in this configuration. If $P=0$, this tension, which is the coefficient of w_{ss} in (4.13), is constant whether or not the string is uniform. This tension need not be positive. Where it is, the equation is hyperbolic; where it is negative, the equation is elliptic. In the latter case, our initial value problem is not well posed. That this is not surprising is apparent from the erratic behavior of a rubber band under compression. This same absence of hyperbolicity (where the tension is negative) can occur in the full nonlinear equations for elastic strings and represents a source of serious technical difficulty. This difficulty can be removed by endowing the string with resistance to bending and twisting (i.e., by replacing the string theory with a rod theory) at the cost of enlarging the number of equations. These enlarged systems have a number of attractive analytic features, which compensate to some extent for their complexity (cf. [2]).

The left side of (4.8) is exactly the Gâteaux differential at $\sigma\mathbf{i}$ in the direction $\mathbf{r}_1$ of the operator $\mathbf{r}\mapsto[N_0(|\mathbf{r}_x|,\cdot)\mathbf{r}_x/|\mathbf{r}_x|]_x-\rho(\cdot)\mathbf{r}_{tt}$ from $C^2([0,1]\times[0,T])$ to $C^0([0,1]\times[0,T])$. For the reasons mentioned at the beginning of Section 3, the domain $C^2([0,1]\times[0,T])$ of this nonlinear operator is singularly inappropriate. Thus, we find ourselves in the paradoxical situation of being unable to reconcile (4.8), which is easy to analyze, which describes the physics of small vibrations with good accuracy, and which is obtained from (2.7), (2.8), (2.11) by a natural but unfortunately purely formal process, with the nonlinear system (2.7), (2.8), (2.11), which is derived in a geometrically exact way from fundamental physical principles and from a general assumption on the material behavior of the string. I know of no work that gives a mathematically precise resolution of this incompatibility, an incompatibility due to the characteristic irregularity of solutions of quasilinear hyperbolic systems.

By examining some related problems, however, it is possible to gain insight into the nature of this difficulty and thereby to be in a position to suggest ways to clarify this issue. Let us first study the quasilinear system (2.7), (2.8), (2.11). If the partial derivative of N_1 with respect to its second argument is positive, then this system, which describes the motion of a viscoelastic string with internal friction, has a parabolic character. This system can be linearized to produce a system like (4.8). If the nonlinear problem is posed in a suitable space of Hölder continuous functions, then the work of **[5]** shows that for small ε, the (unique) solution of (2.1), (2.2), (2.7), (2.8), (2.12) under assumption (4.6) is approximated by (4.7) in the norm of the function space, where $\mathbf{r}_1, \mathbf{r}_2, \ldots$ satisfy linear problems analogous to (4.8). (This proof is based upon an implicit function theorem. For a global analysis of purely longitudinal motion of such viscoelastic problems, see [7].) Many weaker frictional mechanisms, described by (2.10), do not destroy the hyperbolic character of the governing equations (where the tension is positive). The same is true of an elastic string subject to air resistance; this is governed by (2.7), (2.8), (2.11) with $\mathbf{f}$ having the form $-\mathbf{A}\mathbf{r}_t$, where $\mathbf{A}$ is a positive-definite matrix possibly depending on $\mathbf{r}$. The work of **[9]**, **[18]** suggests that for such problems there is a threshold distinguishing "small" from "large" initial conditions; solutions with large initial conditions exhibit shocks, those with small initial conditions do not. It therefore seems possible to relate an amplitude of the frictional effect to the amplitude ε of the initial data so that the equation for $\mathbf{r}_1$ is the vectorial wave equation (4.8), so that the nonlinear problem nevertheless has sufficient frictional dissipation to establish the requisite threshold for initial data, and so that an implicit function theorem can be used in a suitable space to justify (4.7) in a rigorous way. An alternative approach to justifying (4.7) might be fashioned on the observation that the equations for an elastic string should have classical solutions on a time interval aproaching infinity as the initial data approach zero.

We have only exhibited the equations for $\mathbf{r}_1$ of (4.7). A physically illuminating determination of $\mathbf{r}_2$ and $\mathbf{r}_3$ for a string made of a special elastic material was carried out in **[6]**. A general account of such perturbation methods, which describes efficient methods for handling the approximating systems (by means of alternative theorems) and which contains an extensive bibliography, is given in **[12]**.

5. Conclusion. The first steps toward correctly formulated equations for the vibrating string were made by Taylor in 1713. In 1743 d'Alembert derived the first explicit partial differential equation for the small motion of a heavy string. The correct equations for the large vibrations of a string in a plane, equivalent to the planar version of (2.7), were derived by Euler in 1744 by taking the limit of the equations for a discrete model. The correct linear equations for the small planar transverse motion of a string, which is just the wave equation, was obtained and brilliantly analyzed by d'Alembert in 1746. In 1750, Euler stated "Newton's equations of motion" and used them to derive the equations of motion for a string in a manner related to the one we used in Section 2. The spatial equations of motion for strings were obtained by Lagrange in 1761. A critical historical appraisal of these pioneering researches superficially outlined here, with full bibliographic references, is given in **[20]**. The status of the traditional ad hoc assumption that every particle on an elastic string moves in a plane perpendicular to $\mathbf{i}$ and through its position in $\sigma\mathbf{i}$ was clarified by J. B. Keller's **[11]** study of the exact equations. He showed that there is but one material for which the particles execute such a motion with the string having a sinusoidal shape.

The proof that $\mathbf{n}^+ = \mathbf{n}^-$ culminating in (2.6) was obtained by Euler in 1771 and followed earlier work of Pardies in 1673 and Jas. Bernoulli in 1691-1704. The theory of stress, which generalizes (2.6) to three dimensions, was obtained by Cauchy in 1823-1827. A modern generalization of Cauchy's result in a setting that does not rely on his smoothness assumptions is given by **[10]**. Our proof in Section 3 that $\mathbf{n}^{+}=\mathbf{n}^{-}$ (roughly speaking, as measures) generalizes the proof of Section 2 just as the proof of **[10]** generalizes that of Cauchy.

The Principle of Virtual Work in the form commonly used today was laid down by Lagrange in 1788. The proof in Section 3 that a precisely formulated version of this principle is equivalent to precisely formulated Impulse-Momentum Law is roughly modeled on the development of **[3]** for three-dimensional problems of continuous mechanics. The difficulties associated with the nature of boundaries of three-dimensional bodies immerse **[3]** in far deeper questions of analysis and measure theory than those confronted in Section 3. Our entire development can be easily and naturally extended to describe the behavior of rods, which resist bending and which can suffer shear, torsion, and other modes of deformation in addition to the stretching and bending that a string undergoes.

In his derivation of 1744 of the equations of motion of the vibrating string, Euler followed the earlier example of Huygens in 1673 and Joh. Bernoulli in 1727 of regarding the equations as the limit of those for a finite collection of beads joined by massless springs as the number of beads approach infinity while their total mass remains fixed. (Lagrange used a similar approach in 1759 and in part of his work of 1761. Euler's work subsequent to 1760 did not rely on this artifice.) The motion of the system of beads is described by a finite system of ordinary differential equations. It is natural to ask: In what sense does the solution of this or of related systems of ordinary differential equations approximate the solution of the partial differential equations? This question has not been answered for the quasilinear equations of the string. Many of the difficulties for elastic strings are similar to those described in Section 4. Accessible information for viscoelastic strings would come from the modern exploitation of the weak equations (3.11), (2.10) by the Faedo-Galerkin method (cf. **[15]**). An analysis along these lines for a quasilinear engineering model of an elastic string was performed in **[8]**. This work proves the convergence of solutions of a system of ordinary differential equations to the classical solution of the partial differential equation on the time interval before the advent of shocks.

The equations for an elastic string are formally equivalent to the Euler-Lagrange equations for the extremization of the Lagrangian functional and the weak version of these equations is formally equivalent to the vanishing of the first variation (Gâteaux differential) of this functional. The difficulties associated with the irregularity of solutions of the governing equations have so far prevented the exploitation of variational methods for treating these equations. The internal friction of viscoelastic strings means that their motion is not conservative and that their equations do not have a natural variational characterization.

Nonlinear wave equations arising in modern physics have been subjected to an intensive, fruitful, but by no means exhaustive analysis (cf. **[15]**, **[19]**). This analysis is possible because these equations are semilinear and do not have solutions with shocks. These semilinear equations, which are obtained by adding a nonlinear perturbation to the linear wave equation, have the form $u_{tt}-u_{xx}+f(u,u_x)=0$; they should be contrasted with the quasilinear system (2.7), (2.11), in which the coefficients of the highest x-derivatives depend upon the derivative of the unknown function. Thus, the nonlinear wave equations arising from the conceptually simple field of classical continuum mechanics are harder to analyze than those arising from conceptually difficult fields of modern physics.

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36. What we above all things want is, I believe, a varied production of modernized didactic text-books. I have congratulated the Society on the work of recent years, largely inspired by itself, in the production of ambitious treatises calculated to exhibit to the inner circle of accomplished mathematicians a fuller knowledge of recent mathematical advances, calculated to induce those who are already real researchers to research nearer the present confines of known mathematical truth, to give larger views to those who are to lead on coming mathematicians. The next thing is for those whose views are enlarged to do their duty as leaders by endeavouring to secure that the elementary teaching of mathematics be as captivating as ever, but so conveyed that thought be encouraged, that attention to logical soundness in fundamentals be enforced as essential in real mathematics, and by providing lucid and suggestive introductory works on higher subjects, suited to be at once studied by those who have acquired the gift of accurate thought and the possession of elementary knowledge.

— E. B. Elliott, Retiring Presidential Address to the London Mathematical Society, 1898 (Proc. London Math. Soc., (1) 30 (1899) 17).