FINAL EXAM: SOLUTIONS
Math 207B, Winter 2016

1. [20%] Let $J : C^\infty([a,b]) \to \mathbb{R}$ be the functional

$$J(u) = \int_a^b f\left(x, u(x), u'(x)\right)\, dx$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function.
(a) Derive the Euler-Lagrange equation satisfied by stationary points of J.
(b) Show that the Euler-Lagrange equation can be written in the form

$$\frac{d}{dx}\left(f - u' f_{u'}\right) = f_x.$$

Solution.

- (a) This is standard theory. If u is a stationary point of J then for every $\phi \in C_c^\infty(a,b)$

$$\begin{aligned}\left.\frac{d}{d\epsilon}J(u+\epsilon\phi)\right|_{\epsilon=0} &= \int_a^b \left\{f_u(x,u,u')\phi + f_{u'}(x,u,u')\phi'\right\}\, dx \\ &= \int_a^b \left\{f_u(x,u,u') - \frac{d}{dx}f_{u'}(x,u,u')\right\}\phi\, dx \\ &= 0,\end{aligned}$$

  so by the fundamental lemma

$$-\frac{d}{dx}f_{u'} + f_u = 0.$$

  (b) Using the chain rule to expand the derivatives and the Euler-Lagrange equation, we get that

$$\begin{aligned}\frac{d}{dx}\left(f - u' f_{u'}\right) &= f_x + f_u u' + f_{u'} u'' - \left(u'' f_u' + u' \frac{d}{dx} f_{u'}\right) \\ &= f_x + f_u u' + f_{u'} u'' - \left(u'' f_u' + u' f_u\right) \\ &= f_x.\end{aligned}$$

  This form of the Euler-Lagrange equation is sometimes called the duBois-Reymond equation. In particular, if f does not depend explicitly on x, then we have the first integral $f - u' f_{u'} = \text{constant}$.

2. [20%] Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary $\partial\Omega$ and outward unit normal n. Suppose that $u : \overline{\Omega} \to \mathbb{R}$ is the solution of the Neumann problem for the Helmholtz equation

$$\begin{aligned} -\Delta u + u &= 0 \qquad & x \in \Omega, \\ \frac{\partial u}{\partial n} &= f(x) \qquad & x \in \partial\Omega. \end{aligned}$$

Use the formal properties of the δ-function and Green's identity to derive an integral representation for $u(x)$ in terms of the corresponding Green's function $G(x, \xi)$ that satisfies

$$\begin{aligned} -\Delta G + G &= \delta(x - \xi) \qquad & x \in \Omega, \\ \frac{\partial G}{\partial n} &= 0 \qquad & x \in \partial\Omega, \end{aligned}$$

where the Laplacian and the normal derivative are taken with respect to x.

Solution.

- Let $A = -\Delta + 1$ denote the Helmholtz operator. Using Green's second identity, we get for any smooth functions $u, v : \overline{\Omega} \to \mathbb{R}$ that

$$\begin{aligned} \int_\Omega \{uAv - vAu\}\, dx &= \int_\Omega \{u(-\Delta v + v) - v(-\Delta u + u)\}\, dx \\ &= \int_\Omega \{v\Delta u - u\Delta v\}\, dx \\ &= \int_{\partial\Omega} \left\{ v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right\} dS. \end{aligned} \tag{1}$$

  If u, v satisfy homogeneous Neumann conditions, then

$$\int_\Omega uAv\, dx = \int_\Omega vAu\, dx,$$

  so the Neumann Helmholtz operator (with a suitably defined domain) is self-adjoint in $L^2(\Omega)$.

- Taking $v(x) = G(x, \xi)$ in (1) and using the PDEs and boundary conditions satisfied by u and G, we get that

$$\int_\Omega u(x)\delta(x - \xi)\, dx = \int_{\partial\Omega} G(x, \xi) f(x)\, dS(x),$$

so

$$u(\xi) = \int_{\partial\Omega} G(x, \xi) f(x) \, dS(x).$$

- Since A is self-adjoint, the Green's function $G(x, \xi)$ is symmetric, and we can also write this representation as

  $$u(x) = \int_{\partial\Omega} G(x, \xi) f(\xi) \, dS(\xi),$$

  corresponding to the response due to a point-source distribution on the boundary with density f.

3. [30%] Let $0 < k < \pi$, and consider the following BVP:

$$\begin{aligned} -u'' &= k^2 u + f(x), \qquad 0 < x < 1, \\ u'(0) &= A, \qquad u'(1) = B. \end{aligned} \tag{2}$$

(a) Let $G(x, \xi; k)$ be the Green's function that satisfies

$$\begin{aligned} -G_{xx} &= k^2 G + \delta(x - \xi), \qquad 0 < x < 1, \\ G_x(0, \xi; k) &= 0, \qquad G_x(1, \xi; k) = 0. \end{aligned}$$

Give an integral representation of the solution $u(x)$ of (2) in terms of $G(x, \xi; k)$, the function $f(x)$, and the constants A, B.
(b) Compute the Green's function $G(x, \xi; k)$ explicitly. Hint. Note that $\sin(x + y) = \sin x \cos y + \cos x \sin y$.
(c) Let

$$H(x, \xi; k) = G(x, \xi; k) + \frac{1}{k^2}.$$

Show that H has a finite limit as $k \to 0^+$. What is the significance of the limiting function?

Solution.

- Let $L = -d^2/dx^2 - k^2$. By Green's identity, we have

$$\begin{aligned} &\int_0^1 \{u(x) LG(x, \xi; k) - G(x, \xi; k) Lu(x)\}\, dx \\ &\qquad = \int_0^1 \{-u(x) G_{xx}(x, \xi; k) + G(x, \xi; k) u_{xx}(x)\}\, dx \\ &\qquad = [G(x, \xi; k) u_x(x) - u(x) G_x(x, \xi; k)]_{x=0}^{x=1} \end{aligned}$$

  which gives

$$\int_0^1 \{u(x)\delta(x - \xi) - G(x, \xi; k) f(x)\}\, dx = G(1, \xi; k)B - G(0, \xi; k)A,$$

  and therefore

$$u(\xi) = \int_0^1 G(x, \xi; k) f(x)\, dx + G(1, \xi; k)B - G(0, \xi; k)A.$$

- Alternatively, using the symmetry of $G(x, \xi; k)$, we can write the representation as

$$u(x) = \int_0^1 G(x, \xi; k) f(\xi)\, d\xi + G(x, 1; k)B - G(x, 0; k)A.$$

(b) Solving the homogeneous ODE and imposing the Neumann condition at the appropriate end-point, we find that

$$G(x, \xi; k) = \begin{cases} M(\xi) \cos kx & \text{if } 0 \le x < \xi, \\ N(\xi) \cos k(1 - x) & \text{if } \xi < x \le 1 \end{cases}$$

for suitable functions of integration M, N.

- The Green's function $G(x, \xi; k)$ is continuous at $x = \xi$ if

$$M(\xi) = C(\xi) \cos k(1 - \xi), \qquad N(\xi) = C(\xi) \cos k\xi$$

for some function C. The jump condition $-[G_x] = 1$ at $x = \xi$ then gives

$$-C \left\{k \cos k\xi \sin k(1 - \xi) + k \cos k(1 - \xi) \sin k\xi\right\} = 1,$$

or $-Ck \sin k = 1$.

- It follows that

$$G(x, \xi; k) = -\frac{1}{k \sin k} \cos(kx_<) \cos[k(1 - x_>)],$$

where $x_< = \min(x, \xi)$ and $x_> = \max(x, \xi)$.

- (c) Expanding the terms in the Green's function in power series about $k = 0$, we get that

$$\begin{aligned} G(x, \xi; k) &= \frac{-\left[1 - \frac{1}{2}k^2 x_<^2\right]\left[1 - \frac{1}{2}k^2(1 - x_>)^2\right]}{k^2\left(1 - \frac{1}{6}k^2\right)} + O(k^2) \\ &= \frac{-\left[1 - \frac{1}{2}k^2 x_<^2\right]\left[1 - \frac{1}{2}k^2(1 - x_>)^2\right]\left[1 + \frac{1}{6}k^2\right]}{k^2} + O(k^2) \\ &= -\frac{1}{k^2} + \frac{1}{2}x_<^2 + \frac{1}{2}(1 - x_>)^2 - \frac{1}{6} + O(k^2). \end{aligned}$$

It follows that

$$G(x, \xi; k) + \frac{1}{k^2} \to G_M(x, \xi) \qquad \text{as } k \to 0^+$$

where

$$G_M(x, \xi) = \frac{1}{2} x_<^2 + \frac{1}{2}(1 - x_>)^2 - \frac{1}{6}.$$

- The function G_M is the generalized Green's functions for the $k = 0$ problem, as one can verify directly. The term $1/\lambda$, with $\lambda = k^2$, cancels the simple pole in the Green's function $G(x, \xi; \lambda)$ of the Neumann operator $-d^2/dx^2$ at the eigenvalue $\lambda = 0$.

4. [30%] Consider the following eigenvalue problem for a forth-order ODE (using the notation $u^{(4)} = d^4u/dx^4$):

$$u^{(4)} = \lambda u \qquad 0 < x < 1,$$
$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0.$$

(a) Show that the operator $A = d^4/dx^4$ with these boundary conditions is self-adjoint with respect to the standard L^2-inner product, so all eigenvalues λ are real.

(b) Show that every eigenvalue satisfies $\lambda > 0$. Hint. Multiply the ODE by u'' and integrate by parts.

(c) Show that all of the eigenvalues and eigenfunctions are given by $\lambda_n = n^4\pi^4$ and $u_n(x) = \sin(n\pi x)$ where $n = 1, 2, 3, \ldots$.

(d) Use separation of variables to solve the following IBVP for $u(x,t)$:

$$u_t + u_{xxxx} = 0 \qquad 0 < x < 1,\ t > 0$$
$$u(0,t) = u_{xx}(0,t) = 0, \qquad u(1,t) = u_{xx}(1,t) = 0,$$
$$u(x,0) = f(x)$$

How does the solution behave as $t \to \infty$?

Solution.

- (a) Integrating by parts, we get that

$$\begin{aligned}\int_0^1 uAv\,dx &= \int_0^1 uv^{(4)}\,dx \\ &= [uv''']_0^1 - \int_0^1 u'v'''\,dx \\ &= [uv''' - u'v'']_0^1 + \int_0^1 u''v''\,dx.\end{aligned}$$

  Similarly,

$$\int_0^1 vAu\,dx = [vu''' - v'u'']_0^1 + \int_0^1 v''u''\,dx.$$

  It follows that

$$\int_0^1 (uAv - vAu)\,dx = [uv''' - u'v'' + u''v' - u'''v]_0^1$$

- The boundary terms vanish if both u, v satisfy the given homogeneous boundary conditions, so

$$\int_0^1 uAv\,dx = \int_0^1 vAu\,dx,$$

meaning that A is self-adjoint.

- (b) Suppose that u is a nonzero solution of the eigenvalue problem corresponding to an eigenvalue λ. Then

$$\int_0^1 u''u^{(4)}\,dx = \lambda \int_0^1 u''u\,dx.$$

Integration by parts gives

$$\int_0^1 u''u^{(4)}\,dx = [u''u''']_0^1 - \int_0^1 (u''')^2\,dx,$$
$$\int_0^1 u''u\,dx = [u'u]_0^1 - \int_0^1 (u')^2\,dx.$$

The boundary terms vanish, so

$$\lambda = \frac{\int_0^1 (u''')^2\,dx}{\int_0^1 (u')^2\,dx} \geq 0.$$

Note that the denominator is nonzero, since u is a constant if $u' = 0$, and then the boundary conditions imply that $u = 0$.

- If $\int_0^1 (u''')^2\,dx = 0$, then $u''' = 0$, meaning that $u(x) = A + Bx + Cx^2$ is a quadratic function. The boundary conditions at $x = 0$ imply that $A = C = 0$, and then the boundary conditions at $x = 1$ imply that $B = 0$, so $u = 0$. It follows that $\lambda = 0$ is not an eigenvalue and $\lambda > 0$.

- (c) The characteristic equation of the ODE, for solutions $u(x) = e^{rx}$, is $r^4 = \lambda$. Since the eigenvalues λ are real and positive, we can write $\lambda = k^4$ with $k > 0$, and the roots of the characteristic equation are $r = \pm k, \pm ik$. The general solution of the ODE is therefore

$$u(x) = A\cos kx + B\sin kx + C\cosh kx + D\sinh kx.$$

- The boundary conditions at $x = 0$ imply that $A + C = 0$ and $-A + C = 0$, so $A = C = 0$. The boundary conditions at $x = 1$ then imply that

$$B \sin k + D \sinh k = 0, \qquad -B \sin k + D \sinh k = 0.$$

It follows that $D \sinh k = 0$, so $D = 0$ since $\sinh k \neq 0$ for $k \neq 0$. In addition, $B \sin k = 0$, so $B = 0$ unless $\sin k = 0$, which means that $k = n\pi$, $\lambda = n^4\pi^4$, and $u(x) = B \sin(n\pi x)$ with $n = 1, 2, 3, \ldots$.

- (d) Looking for separated solutions $u(x,t) = F(x)G(t)$ of the PDE, we find that

$$\frac{G'}{G} + \frac{F^{(4)}}{F} = 0.$$

Introducing a separation constant λ such that $G'/G = -\lambda$, we get that $G(t) = Ce^{-\lambda t}$ and $F(x)$ is a solution of the eigenvalue problem

$$F^{(4)} = \lambda F, \qquad F(0) = F''(0) = 0, \quad F(1) = F''(1) = 0.$$

From the previous results, $\lambda = n^4\pi^4$ and $F(x) = \sin(n\pi x)$.

- Superposing these separated solutions, we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^4\pi^4 t} \sin(n\pi x).$$

- The initial condition is satisfied if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

so b_n is the Fourier-sine coefficient of f,

$$b_n = 2 \int_0^1 f(x) \sin(n\pi x)\, dx.$$