Midterm: Solutions

Math 207B, Winter 2016

1. Let $p, q, r : [a, b] \to \mathbb{R}$ be smooth functions with $p, r > 0$. Suppose that $u : [a, b] \to \mathbb{R}$ is a smooth extremal over all functions in $C^1([a, b])$ of the functional

$$J(u) = \frac{1}{2} \int_a^b \left\{ p(x) u'^2(x) + q(x) u^2(x) \right\} \, dx$$

subject to the constraint $K(u) = 1$, where

$$K(u) = \frac{1}{2} \int_a^b r(x) u^2(x) \, dx.$$

Show that u is an eigenfunction of the weighted Sturm-Liouville eigenvalue problem

$$\begin{aligned} &-(pu')' + qu = \lambda r u \qquad a < x < b, \\ &u'(a) = 0, \quad u'(b) = 0. \end{aligned}$$

Solution.

- Introduce a Lagrange multiplier λ. Then u is an unconstrained extremal of

  $$I(u, \lambda) = J(u) - \lambda K(u),$$

  and (provided that $\delta K / \delta u \neq 0$, which is the case since $ru \neq 0$)

  $$\frac{d}{d\epsilon} J(u + \epsilon h) \bigg|_{\epsilon = 0} - \lambda \, \frac{d}{d\epsilon} K(u + \epsilon h) \bigg|_{\epsilon = 0} = 0 \qquad \text{for every } h \in C^\infty([a, b]).$$

- Using integration by parts, we get that

  $$\begin{aligned} \frac{d}{d\epsilon} J(u + \epsilon h) \bigg|_{\epsilon = 0} &= \int_a^b \{ pu'h' + quh \} \, dx \\ &= [pu'h]_a^b + \int_a^b \{ -(pu')' + qu \} \, h \, dx, \\ \frac{d}{d\epsilon} K(u + \epsilon h) \bigg|_{\epsilon = 0} &= \int_a^b ruh \, dx. \end{aligned}$$

- It follows that

$$[pu'h]_a^b + \int_a^b \{-(pu')' + qu - \lambda ru\}\, h\, dx = 0 \quad \text{for every } h \in C^\infty([a,b]).$$

- First, considering compactly supported h with $h(a) = 0$, $h(b) = 0$, we get that

$$\int_a^b \{-(pu')' + qu - \lambda ru\}\, h\, dx = 0 \qquad \text{for every } h \in C_c^\infty([a,b]).$$

  Since the integrand is a continuous function, the fundamental lemma of the calculus of variations implies that

$$-(pu')' + qu - \lambda ru = 0 \qquad a < x < b.$$

- It then follows that

$$[pu'h]_a^b = 0 \qquad \text{for every } h \in C^\infty([a,b]).$$

  Choosing h such that $h(a) = 1$, $h(b) = 0$ or $h(a) = 0$, $h(b) = 1$ we find that u satisfies the natural boundary conditions

$$u'(a) = 0, \qquad u'(b) = 0.$$

2. Solve the following IBVP for $u(x,t)$ by the method of separation of variables:

$$\begin{aligned} &u_{tt} - u_{xx} + u = 0, \qquad 0 < x < 1, \quad t > 0 \\ &u_x(0,t) = 0, \quad u_x(1,t) = 0, \\ &u(x,0) = f(x), \qquad u_t(x,0) = 0. \end{aligned}$$

Solution.

- Looking for separated solutions $u(x,t) = F(x)G(t)$, we get that
$$\frac{G''}{G} - \frac{F''}{F} + 1 = 0.$$
Introducing a separation constant $\lambda = -F''/F$, we find that F satisfies the eigenvalue problem
$$-F'' = \lambda F, \qquad F'(0) = 0, \quad F'(1) = 0,$$
and G satisfies the ODE
$$G'' + (\lambda + 1)G = 0.$$

- The eigenvalues and eigenfunctions are
$$\lambda = n^2\pi^2, \qquad F(x) = \cos n\pi x \qquad n = 0, 1, 2, \ldots$$

- The corresponding solution for G is
$$G(t) = A\cos\omega_n t + B\sin\omega_n t \qquad \omega_n = \sqrt{n^2\pi^2 + 1}.$$
Since $u_t = 0$ at $t = 0$, we take $B = 0$.

- Superposing separated solutions, we get that
$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t)\cos(n\pi x).$$
where the coefficients a_n are chosen so that
$$f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x).$$

- By orthogonality, it follows that
$$a_0 = \int_0^1 f(x)\,dx, \qquad a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx \quad n \geq 1.$$

3. Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function and the boundary-value problem

$$-u'' = \pi^2 u + f(x),$$
$$u(0) = 0, \quad u(1) = 0$$

has a solution $u \in C^2([0,1])$. Show that f must satisfy the solvability condition

$$\int_0^1 f(x) \sin(\pi x)\, dx = 0.$$

Solution.

- Multiplying the ODE by $\sin(\pi x)$, integrating over $[0,1]$, and integrating by parts, we get that

$$\begin{aligned}\int_0^1 f \sin(\pi x)\, dx &= -\int_0^1 (u'' + \pi^2 u) \sin(\pi x)\, dx \\ &= [\pi u \cos(\pi x) - u' \sin(\pi x)]_0^1 - \int_0^1 u \left\{ [\sin(\pi x)]'' + \pi^2 \sin(\pi x) \right\} dx \\ &= 0.\end{aligned}$$

  The boundary terms vanish since both u and $\sin(\pi x)$ are zero at $x = 0, 1$.

Remark. In general, if A is a self-adjoint operator on a Hilbert space with eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction ϕ, then a necessary condition for the equation $Au = \lambda u + f$ to have a solution for u is that f is orthogonal to ϕ, since

$$\langle f, \phi \rangle = \langle (A - \lambda I)u, \phi \rangle = \langle u, (A - \lambda I)\phi \rangle = 0.$$

4. Let $A : D(A) \subset L^2(0, 2\pi) \to L^2(0, 2\pi)$ be the operator

$$A = -i\frac{d}{dx} + x,$$

with periodic boundary conditions

$$D(A) = \{u : [0, 2\pi] \to \mathbb{C} : u \in H^1(0, 2\pi),\ u(0) = u(2\pi)\}$$

(a) Show that A is self-adjoint.
(b) Compute the eigenvalues and eigenfunctions of A.

Solution.

- (a) For $u, v \in D(A)$, we have

$$\begin{aligned}
\langle u, Av\rangle &= \int_0^{2\pi} \bar{u}(-iv' + xv)\, dx \\
&= -i\, [\bar{u}v]_0^{2\pi} + \int_0^{2\pi} \{i\bar{u}' + x\bar{u}\}\, v\, dx \\
&= \int_0^{2\pi} \{\overline{-iu' + xu}\}\, v\, dx \\
&= \langle Au, v\rangle
\end{aligned}$$

  The boundary terms vanish since both u and v satisfy periodic boundary conditions.

- (b) The eigenvalue problem for A is

$$-iu' + xu = \lambda u, \qquad u(0) = u(2\pi).$$

  Solving the ODE by separating varaibles, we get that

$$-i\int \frac{du}{u} = \int (\lambda - x)\, dx$$

  so $\log u = i(\lambda x - x^2/2) + C$, and a nonzero solution is

$$u(x) = e^{ix(\lambda - x/2)}.$$

- This function satisfies the periodic boundary conditions if

$$1 = e^{2\pi i(\lambda - \pi)},$$

which means that $\lambda - \pi = n$ for some $n \in \mathbb{Z}$.

- It follows that the eigenvalues $\lambda = \lambda_n$ and eigenfunctions $u = u_n$ are given by

$$\lambda_n = n + \pi, \quad u_n(x) = e^{i(nx + \pi x - x^2/2)} \qquad n \in \mathbb{Z}.$$