PROBLEM SET 1

Math 207B, Winter 2016

Due: Fri., Jan. 22

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x,y) = \frac{xy^3}{x^2 + y^6} \qquad \text{if } (x,y) \neq (0,0).$$

(a) Show that the directional derivatives of f at $(0,0)$ exist in every direction. What is its Gâteaux derivative at $(0,0)$?

(b) Show that f is not Fréchet differentiable at $(0,0)$. (HINT. A Fréchet differentiable function must be continuous.)

2. Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = x^2 + y^2, \qquad g(x,y) = (y-1)^3 - x^2.$$

Find the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = 0$. Show that there does not exist any constant λ such that $\nabla f = \lambda \nabla g$ at some point $(x,y) \in \mathbb{R}^2$. Why does the method of Lagrange multipliers fail in this example?

3. Derive the Euler-Lagrange equation for a functional of the form

$$J(u) = \int_a^b F(x, u, u', u'')\, dx.$$

What are the natural boundary conditions for this functional?

4. A curve $y = u(x)$ with $a \leq x \leq b$, $u(x) > 0$, and $u(a) = u_0$, $u(b) = u_1$ is rotated about the x-axis. Find the curve that minimizes the area of the surface of revolution,

$$J(u) = \int_a^b u\sqrt{1 + (u')^2}\, dx.$$

5. Let X be the space of smooth functions $u : [0, 1] \to \mathbb{R}$ such that $u(0) = 0$, $u(1) = 0$. Define functionals $J, K : X \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_0^1 (u')^2 \, dx, \qquad K(u) = \frac{1}{2} \int_0^1 u^2 \, dx.$$

(a) Introduce a Lagrange multiplier and write down the Euler-Lagrange equation for extremals in X of the functional $J(u)$ subject to the constraint $K(u) = 1$.

(b) Solve the eigenvalue problem in (a) and find all of the extremals. Which one minimizes $J(u)$?

6. (a) Make a change of variable $x = \phi(t)$, $v(t) = u(\phi(t))$, where $\phi'(t) > 0$, in the functional

$$J(u) = \int_a^b F(x, u, u') \, dx.$$

Show that $J(u) = K(v)$ where $K(v)$ has the form

$$K(v) = \int_c^d G(t, v, v') \, dt$$

and express G in terms of F and ϕ.

(b) Show that the Euler-Lagrange equation for $K(v)$ is the same as what you get by changing variables in the Euler-Lagrange equation for $J(u)$.