Problem set 1: Solutions

Math 207B, Winter 2016

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x,y) = \frac{xy^3}{x^2 + y^6} \qquad \text{if } (x,y) \neq (0,0).$$

(a) Show that the directional derivatives of f at $(0,0)$ exist in every direction. What is its Gâteaux derivative at $(0,0)$?

(b) Show that f is not Fréchet differentiable at $(0,0)$. (Hint. A Fréchet differentiable function must be continuous.)

Solution.

- (a) The directional derivative of f at $(0,0)$ in the direction $(h,k) \neq (0,0)$ is

$$\begin{aligned} df(0,0;h,k) &= \frac{d}{d\epsilon} f(\epsilon h, \epsilon k)\bigg|_{\epsilon=0} \\ &= \lim_{\epsilon \to 0} \left(\frac{f(\epsilon h, \epsilon k) - f(0,0)}{\epsilon} \right) \\ &= \lim_{\epsilon \to 0} \left(\frac{\epsilon h k^3}{h^2 + \epsilon^4 k^6} \right) \\ &= 0. \end{aligned}$$

  So all of the directional derivatives exist and $df(0,0;h,k) = 0$.

- (b) If $f : \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable at $\vec{x} \in \mathbb{R}^n$, then it follows directly from the definition that $f(\vec{x} + \vec{h}) \to f(\vec{x})$ as $\vec{h} \to 0$, so f is continuous at $\vec{x}$.

- On the curve $x = t^3$, $y = t$, we have $f(t^3, t) = 1/2$ for $t \neq 0$, so $f(t^3, t) \not\to 0$ as $t \to 0$. It follows that f is not continuous at $(0,0)$ and therefore f is not Fréchet differentiable at $(0,0)$.

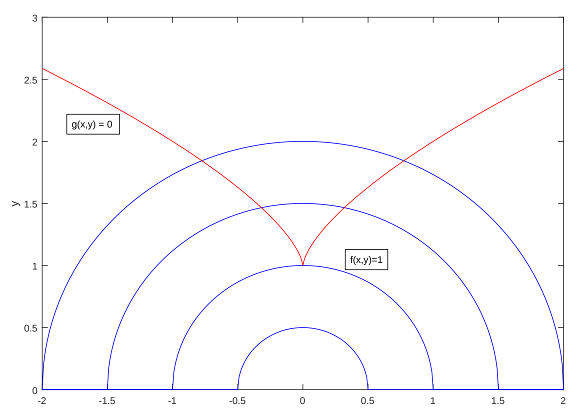
2. Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = x^2 + y^2, \qquad g(x,y) = (y-1)^3 - x^2.$$

Find the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = 0$. Show that there does not exist any constant λ such that $\nabla f = \lambda \nabla g$ at some point $(x,y) \in \mathbb{R}^2$. Why does the method of Lagrange multipliers fail in this example?

Solution.

- On the curve $g(x,y) = 0$, we have $y = 1 + x^{2/3} \geq 1$, so $f(x,y) \geq 1$. On the other hand, $f(0,1) = 1$ and $g(0,1) = 0$, so the minimum value of $f(x,y)$ is 1, attained at $(x,y) = (0,1)$.
- The Lagrange-multiplier-equations $\nabla f = \lambda \nabla g$ are
$$2x = -2\lambda x, \qquad 2y = 3\lambda(y-1)^2.$$
- The first equation is satisfied if either $x = 0$ and λ is arbitrary, or $\lambda = -1$. If $x = 0$, then the constraint $g(x,y) = 0$ implies that $y = 1$, in which case the second equation does not hold for any value of λ.
- On the other hand, if $\lambda = -1$, then $3y^2 - 4y + 3 = 0$, which implies that $y = (2 \pm \sqrt{-5})/3$, so there are no real-valued solutions for y.
- The Lagrange-multiplier method fails because $\nabla g = 0$ at the point $(x,y) = (0,1)$ where f attains its minimum on $g = 0$. As a result, the curve $g(x,y) = 0$ is not smooth with a well-defined normal vector at that point (see figure).



3. Derive the Euler-Lagrange equation for a functional of the form

$$J(u) = \int_a^b F(x, u, u', u'')\,dx.$$

What are the natural boundary conditions for this functional?

Solution.

- Computing the directional derivative of J at u in the direction ϕ, and using integration by parts, we get that

$$\begin{aligned}
dJ(u;\phi) &= \frac{d}{d\epsilon}\int_a^b F(x, u+\epsilon\phi, u'+\epsilon\phi', u''+\epsilon\phi'')\,dx\bigg|_{\epsilon=0} \\
&= \int_a^b \{F_u(x,u,u',u'')\phi + F_{u'}(x,u',u'')\phi' + F_{u''}(x,u,u',u'')\phi''\}\,dx \\
&= \int_a^b \left\{F_u(x,u',u'') - \frac{d}{dx}F_{u'}(x,u',u'') + \frac{d^2}{dx^2}F_{u''}(x,u,u',u'')\right\}\phi\,dx \\
&+ \left[F_{u'}(x,u,u',u'')\phi - \frac{d}{dx}F_{u''}(x,u,u',u'')\cdot\phi + F_{u''}(x,u,u',u'')\phi'\right]_a^b.
\end{aligned}$$

- If u is a smooth extremal of J, then $dJ(u;\phi) = 0$ for every $\phi \in C_c^\infty([a,b])$. In particular, if ϕ and its derivatives are zero at $x = a, b$, then the boundary terms in the integration by parts vanish and

$$\int_a^b \left\{F_u(x,u',u'') - \frac{d}{dx}F_{u'}(x,u',u'') + \frac{d^2}{dx^2}F_{u''}(x,u,u',u'')\right\}\phi\,dx = 0$$

  for all $\phi \in C_c^\infty(a,b)$. The fundamental lemma of the calculus of variations implies that u satisfies the Euler-Lagrange equation

$$\frac{d^2}{dx^2}F_{u''}(x,u,u',u'') - \frac{d}{dx}F_{u'}(x,u',u'') + F_u(x,u',u'') = 0 \qquad a < x < b.$$

- It follows that if $\phi \in C_c^\infty([a,b])$ is non-zero at $x = a, b$, then

$$\left[F_{u'}(x,u,u',u'')\phi - \frac{d}{dx}F_{u''}(x,u,u',u'')\cdot\phi + F_{u''}(x,u,u',u'')\phi'\right]_a^b = 0.$$

Choosing functions ϕ such that only one of $\phi(a)$, $\phi(b)$, $\phi'(a)$, or $\phi'(b)$ is nonzero, we conclude that the natural boundary conditions for u at $x = a, b$ are

$$-\frac{d}{dx}F_{u''}(x, u, u', u'') + F_{u'}(x, u, u', u'') = 0, \quad F_{u''}(x, u, u', u'') = 0.$$

This gives four natural boundary conditions for the Euler-Lagrange equation, which is a fourth-order ODE (provided that $F_{u''u''} \neq 0$).

4. A curve $y = u(x)$ with $a \le x \le b$, $u(x) > 0$, and $u(a) = u_0$, $u(b) = u_1$ is rotated about the x-axis. Find the curve that minimizes the area of the surface of revolution,

$$J(u) = \int_a^b u\sqrt{1 + (u')^2}\, dx.$$

Solution.

- Since the Lagrangian $F(u, u') = u\sqrt{1 + (u')^2}$ is independent of x, the Euler-Lagrange equation for $J(u)$ has the first integral

  $$-u' F_{u'} + F_u = c_1$$

  where c_1 is a constant of integration, which gives

  $$\frac{u}{\sqrt{1 + (u')^2}} = c_1.$$

- The solution for u' is

  $$u' = \sqrt{\frac{u^2}{c_1^2} - 1},$$

  and separation of variables gives

  $$\int \frac{du}{\sqrt{u^2/c_1^2 - 1}} = \int dx.$$

- Making the substitution $u = c_1 \cosh t$, where $(\cosh t)' = \sinh t$ and $\cosh^2 t - \sinh^2 t = 1$, we get that $c_1 t = x + c_2$, so

  $$u(x) = c_1 \cosh\left(\frac{x}{c_1} + c_3\right).$$

- We choose the constants c_1, c_3 so that

  $$u_0 = c_1 \cosh\left(\frac{a}{c_1} + c_3\right), \qquad u_1 = c_1 \cosh\left(\frac{b}{c_1} + c_3\right).$$

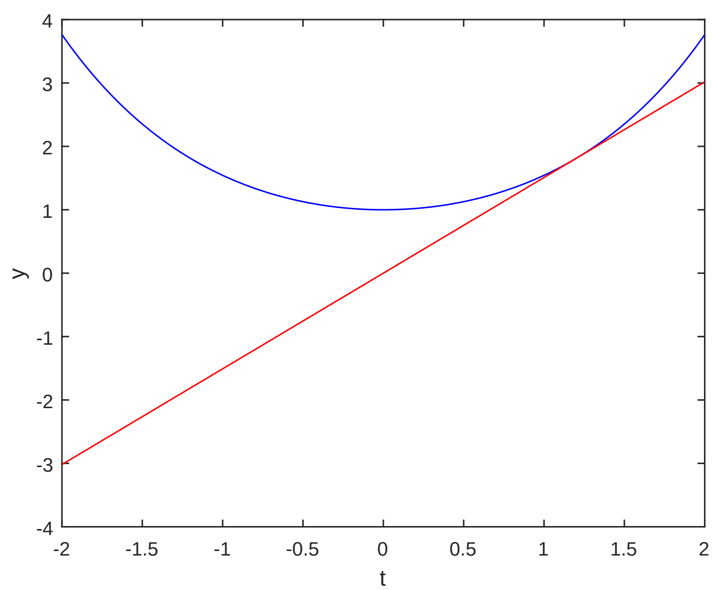
  These algebraic equations might not have a solution for (c_1, c_3), in which case there is no smooth curve that gives a minimal surface of revolution with radius u_0 at $x = a$ and radius u_1 at $x = b$.

- For example, consider the case when $a = -b$, with $b > 0$, and $u_0 = u_1$. Then $c_3 = 0$ and
$$u_0 = c \cosh\left(\frac{b}{c}\right).$$
Writing $y = u_0/c$, $t = b/c$, and $m = u_0/b$, we see that this equation has a solution for c if
$$y = \cosh t, \qquad y = mt.$$

- The line $y = mt$ is tangent to the curve $y = \cosh t$ at $t = t_0$ when $m = m_0$ (see figure), where
$$m_0 t_0 = \cosh t_0, \qquad m_0 = \sinh t_0.$$

- If $0 < m < m_0$, meaning that $u_0 < m_0 b$, then the line $y = mt$ does not intersect the curve $y = \cosh t$, and there are no solutions; if $m > m_0$, meaning that $u_0 > m_0 b$, then the line $y = mt$ intersects $y = \cosh t$ in two points, and there are two solutions.

- The critical value of m is given by $m_0 = \sinh t_0$ where $t_0 > 0$ is the solution of $t_0 \tanh t_0 = 1$. The numerical solution of this transcendental equation is $t_0 \approx 1.1997$, which gives $m_0 \approx 1.5089$.



5. Let X be the space of smooth functions $u : [0,1] \to \mathbb{R}$ such that $u(0) = 0$, $u(1) = 0$. Define functionals $J, K : X \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_0^1 (u')^2\,dx, \qquad K(u) = \frac{1}{2}\int_0^1 u^2\,dx.$$

(a) Introduce a Lagrange multiplier and write down the Euler-Lagrange equation for extremals in X of the functional $J(u)$ subject to the constraint $K(u) = 1$.
(b) Solve the eigenvalue problem in (a) and find all of the extremals. Which one minimizes $J(u)$?

Solution.

- (a) We have
$$\frac{\delta J}{\delta u} = -u'', \qquad \frac{\delta K}{\delta u} = u,$$
so the Lagrange-multiplier equation $\delta J/\delta u = \lambda \delta K/\delta u$ is
$$-u'' = \lambda u, \qquad u(0) = 0, \quad u(1) = 0.$$

- If $\lambda = -k^2 < 0$, then the general solution of the ODE is
$$u(x) = c_1 \cosh x + c_2 \sinh x.$$
The BC $u(0) = 0$ implies that $c_1 = 0$, and then the BC $u(1) = 0$ implies that $c_2 = 0$, so $u = 0$ and it does not satisfy the constraint $K(u) = 1$.

- If $\lambda = 0$, then $u = c_1 + c_2 x$, and the BCs again imply that $u = 0$.

- If $\lambda = k^2 > 0$, then the general solution of the ODE is
$$u(x) = c_1 \cos x + c_2 \sin x.$$
The BC $u(0) = 0$ implies that $c_1 = 0$, and then the BC $u(1) = 0$ is satisfied for $c_2 \neq 0$ if $\sin k = 0$.

- Without loss of generality, we can take $k = n\pi$ with $n = 1, 2, 3, \ldots$, when $u(x) = c \sin(n\pi x)$ and
$$\lambda = n^2\pi^2.$$

- The constraint $K(u) = 1$ is satisfied if

$$\frac{1}{2}c^2 \int_0^1 \sin^2(n\pi x)\, dx = 1,$$

or $c^2/4 = 1$, so the constrained extremals are $u = \pm u_n$ where

$$u_n(x) = 2\sin(n\pi x), \qquad \text{for } n = 1, 2, 3, \dots$$

- We have

$$J(u_n) = \frac{1}{2}\int_0^1 4n^2\pi^2 \cos^2(n\pi x)\, dx = n^2\pi^2.$$

The minimum of J is π^2, attained at u_1. For $n \geq 2$, the extremals u_n are saddle points of J.

6. (a) Make a change of variable $x = \phi(t)$, $v(t) = u(\phi(t))$, where $\phi'(t) > 0$, in the functional

$$J(u) = \int_a^b F(x, u, u')\, dx.$$

Show that $J(u) = K(v)$ where $K(v)$ has the form

$$K(v) = \int_c^d G(t, v, v')\, dt$$

and express G in terms of F and ϕ.

(b) Show that the Euler-Lagrange equation for $K(v)$ is the same as what you get by changing variables in the Euler-Lagrange equation for $J(u)$.

Solution.

- By the chain rule

$$v'(t) = \frac{d}{dt} u(\phi(t)) = \phi'(t) u'(\phi(t)).$$

- Making the change of variables $x = \phi(t)$ in the integral for $J(u)$, we get

$$J(u) = \int_c^d F\left(\phi(t), v(t), v'(t)/\phi'(t)\right)\, \phi'(t) dt$$

where $c = \phi(a)$, $d = \phi(b)$. It follows that

$$G(t, v, v') = \phi'(t) F\left(\phi(t), v, v'/\phi'(t)\right).$$

- The Euler-Lagrange equation for $K(v)$ is

$$-\frac{d}{dt} G_{v'} + G_v = 0.$$

From the previous expression for G, we have $G_{v'} = F_{u'}$ and $G_v = \phi' F_u$, so the Euler-Lagrange equation becomes

$$-\frac{d}{dt} F_{u'} + \phi'(t) F_u = 0.$$

Since

$$\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx} = \phi'(t)\frac{d}{dx},$$

it follows that

$$-\frac{d}{dx}F_{u'} + F_u = 0,$$

which shows that the Euler-Lagrange equation for $K(v)$ is equivalent to the one for $J(u)$.

Remark. The Euler-Lagrange equations are also invariant under more general transformations of the independent and dependent variables. It is often convenient to obtain equations that are invariant under some group of transformations by deriving them from an invariant Lagrangian. For example, in relativistic classical field theory, Lorentz-invariant Lagrangians lead to Lorentz-invariant field equations.