Problem set 2

Math 207B, Winter 2016

Due: Fri, Jan. 29

1. A particle of mass m with position $\vec{x}(t)$ at time t has potential energy $V(\vec{x})$ and kinetic energy

$$T = \frac{1}{2}m|\vec{x}_t|^2.$$

The action of the particle over times $t_0 \le t \le t_1$ is the time-integral of the difference between the kinetic and potential energy:

$$S(\vec{x}) = \int_{t_0}^{t_1} (T - V)\, dt.$$

(a) Show that an extremal $\vec{x}(t)$ of S satisfies Newton's second law $\vec{F} = m\vec{a}$ for motion in a conservative force field $\vec{F} = -\nabla V$.

(b) Show that the total energy of the particle $E = T + V$ is a constant independent of time.

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary (so the divergence theorem holds) and $f : \overline{\Omega} \to \mathbb{R}$ a smooth function. Derive the Euler-Lagrange equation and natural boundary condition that are satisfied by a smooth extremal $u : \overline{\Omega} \to \mathbb{R}$ of the functional

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx.$$

3. The transverse displacement at position x and time t of an elastic string vibrating in the (x, y)-plane is given by $y = u(x, t)$, where $a \le x \le b$ and $t_0 \le t \le t_1$. If the density of the string per unit length is $\rho(x)$ and the tension in the string is a constant T, then (for small displacements) the motion of the string is an extremum of the action

$$S(u) = \int_{t_0}^{t_1} \int_a^b \left(\frac{1}{2}\rho u_t^2 - \frac{1}{2}T u_x^2\right) dxdt.$$

Derive the Euler-Lagrange equation for $u(x, t)$.

4. The (n-dimensional) area of a surface $y = u(x)$ over a region $\Omega \subset \mathbb{R}^n$ is given by

$$J(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

Find the Euler-Lagrange equation (called the minimal surface equation) that is satisfied by a smooth extremum of this functional.

5. Let $X = \{u \in C^1([-1,1]) : u(-1) = -1, u(1) = 1\}$, where $C^1([a,b])$ denotes the space of continuously differentiable functions on $[a,b]$. Define $J : X \to \mathbb{R}$ by

$$J(u) = \int_{-1}^{1} x^4 (u')^2\, dx.$$

(a) Show that

$$\inf_{u \in X} J(u) = 0,$$

but $J(u) > 0$ for every $u \in X$ (so J does not attain its infimum on X).

(b) What happens when you try to solve the Euler-Lagrange equation for extremals of J?