Problem set 2: Solutions
Math 207B, Winter 2016

1. A particle of mass m with position $\vec{x}(t)$ at time t has potential energy $V(\vec{x})$ and kinetic energy

$$T = \frac{1}{2} m |\vec{x}_t|^2.$$

The action of the particle over times $t_0 \le t \le t_1$ is the time-integral of the difference between the kinetic and potential energy:

$$S(\vec{x}) = \int_{t_0}^{t_1} (T - V)\, dt.$$

(a) Show that an extremal $\vec{x}(t)$ of S satisfies Newton's second law $\vec{F} = m\vec{a}$ for motion in a conservative force field $\vec{F} = -\nabla V$.
(b) Show that the total energy of the particle $E = T + V$ is a constant independent of time.

Solution.

- (a) The Euler-Lagrange equation for the action

$$S(\vec{x}) = \int_{t_0}^{t_1} L(\vec{x}, \vec{x}_t)\, dt, \qquad L(\vec{x}, \vec{x}_t) = \frac{1}{2} m |\vec{x}_t|^2 - V(\vec{x})$$

is given by

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{x}_t}\right) + \frac{\partial L}{\partial \vec{x}} = 0.$$

- Using the summation convention, and the Kronecker-δ defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

we have

$$\left[\frac{\partial |\vec{a}|^2}{\partial \vec{a}}\right]_i = \frac{\partial}{\partial a_i}(a_j a_j) = 2a_j \delta_{ij} = 2a_i = [2\vec{a}]_i,$$

so $\partial F/\partial \vec{x}_t = m\vec{x}_t$, and $\partial F/\partial \vec{x} = -\nabla V(\vec{x})$.

- The Euler-Lagrange equation is

$$m\vec{x}_{tt} = -\nabla V(\vec{x}),$$

which is Newton's second law.

- (b) Taking the scalar product of the ODE with $\vec{x}_t$, we get that

$$m\vec{x}_t \cdot \vec{x}_{tt} = -\vec{x}_t \cdot \nabla V(\vec{x}).$$

It follows that

$$\frac{d}{dt}\left[\frac{1}{2}m\vec{x}_t \cdot \vec{x}_t + V(\vec{x})\right] = 0,$$

so

$$\frac{1}{2}m|\vec{x}_t|^2 + V(\vec{x}) = \text{constant}.$$

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary (so the divergence theorem holds) and $f : \overline{\Omega} \to \mathbb{R}$ a smooth function. Derive the Euler-Lagrange equation and natural boundary condition that are satisfied by a smooth extremal $u : \overline{\Omega} \to \mathbb{R}$ of the functional

$$J(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - fu \right) \, dx.$$

Solution.

- For $\phi \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned}
&\frac{d}{d\epsilon} J(u + \epsilon\phi)\big|_{\epsilon=0} \\
&\quad = \frac{d}{d\epsilon} \int_\Omega \left[\frac{1}{2} (\nabla u + \epsilon \nabla \phi) \cdot (\nabla u + \epsilon \nabla \phi) - f(u + \epsilon\phi) \right] \, dx \bigg|_{\epsilon=0} \\
&\quad = \int_\Omega (\nabla u \cdot \nabla \phi - f\phi) \, dx
\end{aligned}$$

- The divergence theorem implies that

$$\begin{aligned}
\int_\Omega (\nabla u \cdot \nabla \phi) \, dx &= \int_\Omega [\nabla \cdot (\phi \nabla u) - \phi \Delta u] \, dx \\
&= \int_{\partial\Omega} \phi \frac{\partial u}{\partial n} \, dS - \int_\Omega \phi \Delta u \, dx.
\end{aligned}$$

- If $\phi = 0$ on $\partial\Omega$, then

$$\frac{d}{d\epsilon} J(u + \epsilon\phi)\big|_{\epsilon=0} = -\int_\Omega (\Delta u + f) \, \phi \, dx.$$

  The fundamental lemma of the calculus of variations implies that an extremal u, with

$$\frac{d}{d\epsilon} J(u + \epsilon\phi)\big|_{\epsilon=0} = 0,$$

  for all ϕ satisfies the Euler-Lagrange equation

$$-\Delta u = f \qquad \text{in } \Omega.$$

- For functions ϕ that do not necessarily vanish on the boundary, we have
$$\frac{d}{d\epsilon} J(u+\epsilon\phi)\big|_{\epsilon=0} = \int_{\partial\Omega} \phi \frac{\partial u}{\partial n}\, dS.$$
It follows that the natural boundary condition for an extremal u is the Neumann condition
$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

3. The transverse displacement at position x and time t of an elastic string vibrating in the (x, y)-plane is given by $y = u(x, t)$, where $a \leq x \leq b$ and $t_0 \leq t \leq t_1$. If the density of the string per unit length is $\rho(x)$ and the tension in the string is a constant T, then (for small displacements) the motion of the string is an extremum of the action

$$S(u) = \int_{t_0}^{t_1} \int_a^b \left(\frac{1}{2} \rho u_t^2 - \frac{1}{2} T u_x^2 \right) dx dt.$$

Derive the Euler-Lagrange equation for $u(x, t)$.

Solution.

- For every $\phi \in C_c^\infty((a, b) \times (t_0, t_1))$, we have

$$\frac{d}{d\epsilon} S(u + \epsilon\phi)\Big|_{\epsilon=0} = \int_{t_0}^{t_1} \int_a^b (\rho u_t \phi_t - T u_x \phi_x) \, dx dt.$$

- Integrating by parts and using the fact that $\phi = 0$ at $x = a, b$ and $t = t_0, t_1$, we get that

$$\frac{d}{d\epsilon} S(u + \epsilon\phi)\Big|_{\epsilon=0} = \int_{t_0}^{t_1} \int_a^b (-\rho u_{tt} + T u_{xx}) \, \phi \, dx dt.$$

- The fundamental lemma of the calculus of variations then implies that a smooth extremal u of S satisfies the wave equation

$$\rho u_{tt} = T u_{xx}.$$

4. The (n-dimensional) area of a surface $y = u(x)$ over a region $\Omega \subset \mathbb{R}^n$ is given by

$$J(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

Find the Euler-Lagrange equation (called the minimal surface equation) that is satisfied by a smooth extremum of this functional.

Solution.

- Using the divergence theorem, we find for every $\phi \in C_c^\infty(\Omega)$ that

$$\begin{aligned}
\frac{d}{d\epsilon} J(u + \epsilon\phi)\big|_{\epsilon=0} &= \frac{d}{d\epsilon} \int_\Omega \sqrt{1 + |\nabla u + \epsilon \nabla\phi|^2}\, dx \Bigg|_{\epsilon=0} \\
&= \int_\Omega \frac{1}{2\sqrt{1 + |\nabla u|^2}} \frac{d}{d\epsilon} \left|\nabla u + \epsilon \nabla \phi\right|^2\Big|_{\epsilon=0}\, dx \\
&= \int_\Omega \frac{\nabla u \cdot \nabla \phi}{\sqrt{1 + |\nabla u|^2}}\, dx \\
&= -\int_\Omega \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \phi\, dx.
\end{aligned}$$

- The Euler-Lagrange equation for smooth extremals of J is therefore

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0$$

- Using the summation convention, we can write this equation in component form as

$$\left(\frac{u_{x_i}}{\sqrt{1 + u_{x_j} u_{x_j}}}\right)_{x_i} = 0.$$

5. Let $X = \{u \in C^1([-1,1]) : u(-1) = -1, u(1) = 1\}$, where $C^1([a,b])$ denotes the space of continuously differentiable functions on $[a,b]$. Define $J : X \to \mathbb{R}$ by

$$J(u) = \int_{-1}^{1} x^4 (u')^2 \, dx.$$

(a) Show that

$$\inf_{u \in X} J(u) = 0,$$

but $J(u) > 0$ for every $u \in X$ (so J does not attain its infimum on X).

(b) What happens when you try to solve the Euler-Lagrange equation for extremals of J?

Solution.

- We have $J(u) \geq 0$ for every $u \in X$, so $\inf J(u) \geq 0$.
- To make $J(u)$ as small as possible, we want to make u' small except near $x = 0$, where u' can be large.
- In order to do this, let $f : \mathbb{R} \to \mathbb{R}$ be any continuously differentiable, odd function such that $f(x) \to 1$ as $x \to \infty$ and

$$\int_0^\infty x^4 f'(x)^2 \, dx = C < \infty.$$

  For example, we could choose $f(x) = \tanh x$.
- If we were to include piecewise smooth functions in X, or functions in a Sobolev space such as $H^1(-1,1)$, we could instead take

$$f(x) = \begin{cases} -1 & \text{if } -\infty < x \leq -1, \\ x & \text{if } -1 < x < 1, \\ 1 & \text{if } 1 \leq x < \infty. \end{cases}$$

- For $\epsilon > 0$, define

$$u^\epsilon(x) = \frac{f(x/\epsilon)}{f(1/\epsilon)}.$$

  Then $u^\epsilon(-1) = -1$ and $u^\epsilon(1) = 1$ so $u^\epsilon \in X$.

- Making the change of variables $x = \epsilon t$, we have

$$\begin{aligned} J(u^\epsilon) &= \frac{2}{\epsilon^2 f(1/\epsilon)^2} \int_0^1 x^4 \left[f'\left(\frac{x}{\epsilon}\right)\right]^2 dx \\ &= \frac{2\epsilon^3}{f(1/\epsilon)^2} \int_0^{1/\epsilon} t^4 f'(t)^2 \, dt \\ &\leq \frac{2C\epsilon^3}{f(1/\epsilon)^2}. \end{aligned}$$

  It follows that $J(u^\epsilon) \to 0$ as $\epsilon \to 0^+$, so

$$\inf_{u \in X} J(u) = 0.$$

- If $J(u) = 0$ and $u \in C^1([-1,1])$, then $x^4(u')^2 = 0$, so $u'(x) = 0$ except at $x = 0$, and then $u'(0) = 0$ by continuity. It follows that $u =$ constant, but no constant function satisfies both boundary conditions for functions in X, so $J(u) > 0$ for every $u \in X$.

- (b) The Euler-Lagrange equation for $J(u)$ is

$$-2\left(x^4 u'\right)' = 0.$$

  Integrating this ODE once, we get that $x^4 u' = c$, where c is a constant of integration. Integrating again, we get that

$$u(x) = c_1 + \frac{c_2}{x^3}$$

  where c_1, c_2 are constants of integration.

- The boundary conditions imply that $c_1 - c_2 = -1$, $c_1 + c_2 = 1$, so $c_1 = 0$, $c_2 = 1$.

- The solution of the Euler-Lagrange equation is $u(x) = 1/x^3$, which is singular at $x = 0$, so there is no smooth solution.

Remark. The Lagrangian for $J(u)$ is $F(x, u') = x^4(u')^2$. When $x \neq 0$, the Lagrangian is a strictly convex function of u' (with $F_{u'u'} = 2x^4 > 0$), but strict convexity is lost at $x = 0$. The nonexistence of a minimizer of $J(u)$ is associated with this loss of strict convexity.