PROBLEM SET 3
Math 207B, Winter 2016
Due: Fri, Feb. 5

1. Suppose that $u(x)$ is a non-zero solution of the eigenvalue problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u(0) &= 0, \qquad u(1) = 0. \end{aligned}$$

Show that

$$\lambda = \frac{\int_0^1 (u')^2 \, dx}{\int_0^1 u^2 \, dx}.$$

Deduce that every eigenvalue λ is strictly positive.

2. Heat flows in a rod of length L with a heat source ($a > 0$) or sink ($a < 0$) whose density au is proportional to the temperature u. Suppose that $u(x,t)$ satisfies the IBVP

$$\begin{aligned} &u_t = Du_{xx} + au \qquad 0 < x < L, \quad t > 0, \\ &u(0,t) = 0, \qquad u(L,t) = 0, \\ &u(x,0) = f(x). \end{aligned}$$

(a) Nondimensionalize the problem, and show that the IBVP can be written in nondimensional form as

$$\begin{aligned} &u_t = u_{xx} + \alpha u \qquad 0 < x < 1, \quad t > 0, \\ &u(0,t) = 0, \qquad u(1,t) = 0 \qquad t > 0, \\ &u(x,0) = f(x) \qquad 0 < x < 1, \end{aligned}$$

where α is a suitable nondimensional parameter. Give a physical interpretation of α.

(b) Solve the IBVP in (a) by the method of separation of variables.

(c) How does your solution behave as $t \to \infty$? For what values of α does $u(x,t) \to 0$ as $t \to \infty$? What happens for larger values of α? Give a physical explanation of this behavior in terms of the thermal energy.

3. Solve the following eigenvalue problem for the linear operator $-d^2/dx^2$ with Neumann BCs:

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u'(0) &= 0, \qquad u'(1) = 0. \end{aligned}$$

(a) Find the eigenvalues $\lambda = \lambda_n$, where $n = 0, 1, 2, 3, \dots$, and the corresponding eigenfunctions $u_n(x)$.

(b) Show that the eigenfunctions can be normalized so that

$$\int_0^1 u_m(x) u_n(x)\, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

(c) Does your argument in Problem 1 that $\lambda \neq 0$ work in this case?

4. (a) Solve the following IBVP by the method of separation of variables

$$\begin{aligned} &u_t = u_{xx} && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = 0, && u_x(1,t) = 0 \qquad t > 0, \\ &u(x,0) = f(x) && 0 < x < 1. \end{aligned}$$

(b) How does your solution behave as $t \to \infty$?

(c) Show directly from the IBVP in (a) that

$$\int_0^1 u(x,t)\, dx = \int_0^1 f(x)\, dx \qquad \text{for all } t \geq 0.$$

Is this result consistent with your answer in (b)? Give a physical explanation of the long-time behavior of $u(x,t)$.