Problem set 3: Solutions
Math 207B, Winter 2016

1. Suppose that $u(x)$ is a non-zero solution of the eigenvalue problem

$$-u'' = \lambda u \qquad 0 < x < 1,$$
$$u(0) = 0, \qquad u(1) = 0.$$

Show that

$$\lambda = \frac{\int_0^1 (u')^2\,dx}{\int_0^1 u^2\,dx}.$$

Deduce that every eigenvalue λ is strictly positive.

Solution.

- Multiplying the ODE by u and integrating the result over $[0,1]$, we get that
$$-\int_0^1 uu''\,dx = \lambda\int_0^1 u^2\,dx.$$
Integrating by parts and using the boundary conditions, we have
$$-\int_0^1 uu''\,dx = -\left[uu'\right]_0^1 + \int_0^1 (u')^2\,dx = \int_0^1 (u')^2\,dx,$$
and the result follows.

- The expression for λ shows that $\lambda \geq 0$. Moreover, if $\lambda = 0$, then
$$\int_0^1 (u')^2\,dx = 0$$
so $u' = 0$ (assuming that u' is continuous) and therefore $u = \text{constant}$. The boundary conditions then imply that $u = 0$, so we must have $\lambda > 0$.

- To explain the previous calculation in more general terms, we can write the positive, self-adjoint operator $A = -d^2/dx^2$ as $A = D^*D$ where the skew-adjoint operator $D = d/dx$ with Dirichlet boundary conditions has L^2-adjoint $D^* = -d/dx$. If $Au = \lambda u$, then
$$\lambda = \frac{\langle u, Au\rangle}{\langle u, u\rangle}, \qquad \langle u, Au\rangle = \langle u, D^*Du\rangle = \langle Du, Du\rangle \geq 0,$$
and $\lambda = 0$ only if $Du = 0$.

2. Heat flows in a rod of length L with a heat source ($a > 0$) or sink ($a < 0$) whose density au is proportional to the temperature u. Suppose that $u(x,t)$ satisfies the IBVP

$$\begin{aligned} &u_t = Du_{xx} + au \qquad 0 < x < L, \quad t > 0, \\ &u(0,t) = 0, \qquad u(L,t) = 0, \\ &u(x,0) = f(x). \end{aligned}$$

(a) Nondimensionalize the problem, and show that the IBVP can be written in nondimensional form as

$$\begin{aligned} &u_t = u_{xx} + \alpha u \qquad 0 < x < 1, \quad t > 0, \\ &u(0,t) = 0, \qquad u(1,t) = 0 \qquad t > 0, \\ &u(x,0) = f(x) \qquad 0 < x < 1, \end{aligned}$$

where α is a suitable nondimensional parameter. Give a physical interpretation of α.

(b) Solve the IBVP in (a) by the method of separation of variables.

(c) How does your solution behave as $t \to \infty$? For what values of α does $u(x,t) \to 0$ as $t \to \infty$? What happens for larger values of α? Give a physical explanation of this behavior in terms of the thermal energy.

Solution.

- (a) Let Θ be a typical value of the initial data $f(x)$, and define

$$\bar{x} = \frac{x}{L}, \qquad \bar{t} = \frac{Dt}{L^2}, \qquad u(x,t) = \Theta \bar{u}(\bar{x}, \bar{t}).$$

  Then

$$\partial_x = \frac{1}{L}\partial_{\bar{x}}, \qquad \partial_t = \frac{D}{L^2}\partial_{\bar{t}}.$$

  After changing variables, we get the nondimensionalized problem with a nondimensionalized initial condition and source-parameter

$$\bar{u}(\bar{x}, 0) = \frac{f(L\bar{x})}{\Theta}, \qquad \alpha = \frac{aL^2}{D}.$$

- In the absence of diffusion, a typical time-scale for the source term ($u_t = au$) to lead to exponential growth or decay by a factor of e is $T_s = 1/|a|$. A typical time scale for heat to diffuse the length of the rod is $T_d = L^2/D$, so $|\alpha| = T_d/T_s$ measures the relative speed of diffusion and growth or decay due to the source ($|\alpha| \ll 1$ means that diffusion dominates and $|\alpha| \gg 1$ means that the source term dominates).

- (b) Looking for separated solutions
$$u(x,t) = F(x)G(t)$$
of the PDE, we find that
$$\frac{G'}{G} = \frac{F''}{F} + \alpha.$$
Defining a separation constant λ by $F''/F = -\lambda$ (other definitions would lead to the same final result), we get that
$$G' = (\alpha - \lambda)G, \qquad -F'' = \lambda F.$$
The solution for G is
$$G(t) = G_0 e^{(\alpha-\lambda)t}.$$

- Imposing the boundary conditions on F, we get the eigenvalue problem
$$-F'' = \lambda F, \qquad F(0) = 0, \quad F(1) = 0$$
whose solutions (up to a constant factor in F) are
$$\lambda = n^2\pi^2, \qquad F(x) = \sin(n\pi x) \qquad n = 1, 2, 3, \ldots$$

- Superposing the separated solutions, we get that
$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{(\alpha-n^2\pi^2)t} \sin(n\pi x).$$

- The initial condition is satisfied if
$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$
which gives by orthogonality that
$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

- (c) The solution decays to zero as $t \to \infty$ if $\alpha < \pi^2$ and is unbounded if $\alpha > \pi^2$ (provided that $b_1 \neq 0$). If $\alpha = \pi^2$, then the solution approaches a steady state $u(x,t) \to b_1 \sin(\pi x)$.

- Suppose for definiteness that $u > 0$ inside the rod. If $\alpha < \pi^2$, then heat energy diffuses out of the ends of the rod at a faster rate than its generation by the source inside the rod, and the temperature decays to zero. If $\alpha > \pi^2$, then the source generates heat faster than it can escape and the temperature increases. If $\alpha = \pi^2$ (and $b_1 > 0$), then (for large times) the loss of heat through the ends of the rod balances the generation of heat inside.

3. Solve the following eigenvalue problem for the linear operator $-d^2/dx^2$ with Neumann BCs:

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1, \\ u'(0) &= 0, \qquad u'(1) = 0. \end{aligned}$$

(a) Find the eigenvalues $\lambda = \lambda_n$, where $n = 0, 1, 2, 3, \dots$, and the corresponding eigenfunctions $u_n(x)$.
(b) Show that the eigenfunctions can be normalized so that

$$\int_0^1 u_m(x) u_n(x)\, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

(c) Does your argument in Problem 1 that $\lambda \neq 0$ work in this case?

Solution.

- (a) Since the problem is self-adjoint, all eigenvalues are real.
- If $\lambda = -k^2 < 0$, then the general solution of the ODE is

  $$u(x) = c_1 \cosh kx + c_2 \sinh kx.$$

  The boundary condition $u'(0) = 0$ implies that $c_2 = 0$ and then the boundary condition $u'(1) = 0$ implies that $kc_1 \sinh k = 0$, so $c_1 = 0$ and $u = 0$, meaning that any $\lambda < 0$ is not an eigenvalue.
- If $\lambda = 0$, then

  $$u(x) = c_1 + c_2 x.$$

  The boundary condition $u'(0) = 0$ implies that $c_2 = 0$, and then the boundary condition $u'(1) = 0$ is satisfied for any c_1, so $\lambda = 0$ is an eigenvalue with eigenfunction $u = 1$.
- If $\lambda = k^2 > 0$, then the general solution of the ODE is

  $$u(x) = c_1 \cos kx + c_2 \sin kx.$$

  The boundary condition $u'(0) = 0$ implies that $c_2 = 0$ and then the boundary condition $u'(1) = 0$ implies that $kc_1 \sin k = 0$, so $c_1 = 0$ unless $k = n\pi$ for some $n \in \mathbb{N}$, and $\lambda_n = n^2\pi^2$ is an eigenvalue with eigenfunction $u_n(x) = \cos(n\pi x)$.

- (b) From the addition formula for cosines, we have

$$\cos(m\pi x)\cos(n\pi x)\,dx = \frac{1}{2}\left[\cos(m+n)\pi x + \cos(m-n)\pi x\right].$$

If $m \neq n$, then

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)\pi x}{(m+n)\pi} + \frac{\sin(m-n)\pi x}{(m-n)\pi}\right]_0^1 = 0,$$

and if $m = n$, then

$$\int_0^1 \cos^2(n\pi x)\,dx = \begin{cases} 1 & \text{if } n = 0, \\ 1/2 & \text{if } n \geq 1. \end{cases}$$

- It follows that an orthonormal set of eigenfunctions is given by

$$u_0(x) = 1, \qquad u_n(x) = \sqrt{2}\cos(n\pi x) \quad n \geq 1.$$

(c) The argument is the same up to the conclusion that $u' = 0$ if $\lambda = 0$. In this case, however, a nonzero constant function $u = 1$ satisfies the Neumann boundary conditions, so $\lambda = 0$ is an eigenvalue.

4. (a) Solve the following IBVP by the method of separation of variables

$$\begin{aligned} &u_t = u_{xx} && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = 0, && u_x(1,t) = 0 \qquad t > 0, \\ &u(x,0) = f(x) && 0 < x < 1. \end{aligned}$$

(b) How does your solution behave as $t \to \infty$?

(c) Show directly from the IBVP in (a) that

$$\int_0^1 u(x,t)\,dx = \int_0^1 f(x)\,dx \qquad \text{for all } t \geq 0.$$

Is this result consistent with your answer in (b)? Give a physical explanation of the long-time behavior of $u(x,t)$.

Solution.

- (a) If $u(x,t) = F(x)G(t)$ is a separated solution of the PDE, then

$$\frac{F''}{F} = \frac{G'}{G} = -\lambda,$$

where λ is a separation constant.

- It follows that $G(t) = G_0 e^{-\lambda t}$ and F satisfies the eigenvalue problem

$$-F'' = \lambda F, \qquad F'(0) = 0, \quad F'(1) = 0,$$

whose eigenvalues and eigenfunctions are found in the previous question. The separated solutions are therefore

$$u(x,t) = e^{-n^2\pi^2 t}\cos(n\pi x), \qquad n = 0, 1, 2, 3, \ldots$$

- Superposing the separated solutions, we find that the general solution of the PDE and the BCs is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x).$$

The IC is satisfied if we choose the a_n such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

and orthogonality of the eigenfunctions implies that

$$a_0 = \int_0^1 f(x)\,dx, \qquad a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx \quad \text{for } n \geq 1.$$

- (b) As $t \to \infty$, the temperature $u(x,t)$ approaches a constant a_0.
- (c) Integrating the PDE over $0 \leq x \leq 1$ and using the boundary conditions, we get that

  $$\frac{d}{dt}\int_0^1 u\,dx = \int_0^1 u_t\,dx = \int_0^1 u_{xx}\,dx = [u_x]_{x=0}^{x=1} = 0,$$

  so $\int_0^1 u(x,t)\,dx$ is independent of time and is equal to its initial value $a_0 = \int_0^1 f\,dx$, which is consistent with the solution obtained by separation of variables.
- As $t \to \infty$, diffusion evens out any temperature variations, so the temperature approaches a constant, uniform state. The ends (and sides) of the rod are insulated, with zero heat flux $-u_x$, so no heat energy can escape, and the final constant value of the temperature is the one with the same energy as the initial state (equal to the average value of the initial data f).