Problem set 4
Math 207B, Winter 2016
Due: Fri, Feb. 12

1. The following nonhomogeneous IBVP describes heat flow in a rod whose ends are held at temperatures u_0, u_1:

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = u_0, \qquad u(1,t) = u_1 \\ &u(x,0) = f(x) \end{aligned}$$

(a) Find the steady state temperature $U(x)$ that satisfies

$$\begin{aligned} &U_{xx} = 0 \qquad 0 < x < 1 \\ &U(0) = u_0, \qquad U(1) = u_1 \end{aligned}$$

(b) Write $u(x,t) = U(x) + v(x,t)$ and find the corresponding IBVP for v. Use separation of variables to solve for v and hence u.

(c) How does $u(x,t)$ behave as $t \to \infty$?

2. Define a first-order differential operator with complex coefficients acting in $L^2(0, 2\pi)$ by

$$A = -i\frac{d}{dx}.$$

(a) Show that A is formally self-adjoint.

(b) Show that A with periodic boundary conditions $u(0) = u(2\pi)$ is self-adjoint, and find the eigenvalues and eigenfunctions of the corresponding eigenvalue problem

$$-iu' = \lambda u, \qquad u(0) = u(2\pi).$$

(c) What are the adjoint boundary conditions to the Dirichlet condition $u(0) = 0$ at $x = 0$? Is A with this Dirichlet boundary condition self-adjoint? Find all eigenvalues and eigenfunctions of the corresponding eigenvalue problem

$$-iu' = \lambda u, \qquad u(0) = 0.$$

How does your result compare with the properties of finite-dimensional eigenvalue problems for matrices?

3. Let A be a regular Sturm-Liouville operator, given by

$$Au = -(pu')' + qu,$$

acting in $L^2(a,b)$. Verify that A with the Robin boundary conditions

$$\alpha u'(a) + u(a) = 0, \qquad u'(b) + \beta u(b) = 0$$

is self-adjoint.

4. Show that the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} -u'' &= \lambda u \qquad 0 < x < 1 \\ u(0) &= 0, \qquad u'(1) + \beta u(1) = 0 \end{aligned}$$

are given by $\lambda = k^2$ where $k > 0$ satisfies the equation

$$\beta \tan k + k = 0.$$

Show graphically that there is a infinite sequence of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$ with $\lambda_n \to \infty$ as $n \to \infty$. What is the asymptotic behavior of λ_n as $n \to \infty$?

5. The following IBVP describes heat flow in a rod whose left end is held at temperature 0 and whose right end loses heat to the surroundings according to Newton's law of cooling (heat flux is proportional to the temperature difference):

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \qquad u_x(1,t) = -\beta u(1,t) \\ &u(x,0) = f(x) \end{aligned}$$

Solve this IBVP by the method of separation of variables.