PROBLEM SET 4: SOLUTIONS
Math 207B, Winter2016

1. The following nonhomogeneous IBVP describes heat flow in a rod whose ends are held at temperatures u_0, u_1:

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = u_0, \qquad u(1,t) = u_1 \\ &u(x,0) = f(x) \end{aligned}$$

(a) Find the steady state temperature $U(x)$ that satisfies

$$\begin{aligned} &U_{xx} = 0 \qquad 0 < x < 1 \\ &U(0) = u_0, \qquad U(1) = u_1 \end{aligned}$$

(b) Write $u(x,t) = U(x) + v(x,t)$ and find the corresponding IBVP for v. Use separation of variables to solve for v and hence u.
(c) How does $u(x,t)$ behave as $t \to \infty$?

Solution.

- (a) The solution is a linear function

$$U(x) = u_0 + (u_1 - u_0)x.$$

- (b) The perturbation $v(x,t)$ from the steady state satisfies the IBVP with homogeneous BCs

$$\begin{aligned} &v_t = v_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &v(0,t) = 0, \qquad v(1,t) = 0 \\ &v(x,0) = g(x) \end{aligned}$$

  where

$$g(x) = f(x) - U(x).$$

- The solution is

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

  where

$$b_n = 2\int_0^1 g(x)\sin(n\pi x)\,dx.$$

- (c) As $t \to \infty$, we have $v(x,t) \to 0$ and $u(x,t) \to U(x)$.

2. Define a first-order differential operator with complex coefficients acting in $L^2(0, 2\pi)$ by

$$A = -i\frac{d}{dx}.$$

(a) Show that A is formally self-adjoint.

(b) Show that A with periodic boundary conditions $u(0) = u(2\pi)$ is self-adjoint, and find the eigenvalues and eigenfunctions of the corresponding eigenvalue problem

$$-iu' = \lambda u, \qquad u(0) = u(2\pi).$$

(c) What are the adjoint boundary conditions to the Dirichlet condition $u(0) = 0$ at $x = 0$? Is A with this Dirichlet boundary condition self-adjoint? Find all eigenvalues and eigenfunctions of the corresponding eigenvalue problem

$$-iu' = \lambda u, \qquad u(0) = 0.$$

How does your result compare with the properties of finite-dimensional eigenvalue problems for matrices?

Solution.

- (a) If $u, v \in C^1([0, 2\pi])$, then an integration by parts gives

$$\begin{aligned}
\langle u, Av \rangle &= \int_0^{2\pi} \overline{u(x)} \left[-iv'(x)\right] \, dx \\
&= -i \left[\bar{u}v\right]_0^{2\pi} + \int_0^{2\pi} \overline{\left[-iu'(x)\right]} v(x) \, dx \\
&= -i \left[\bar{u}v\right]_0^{2\pi} + \langle Au, v \rangle,
\end{aligned}$$

which shows that A is formally self-adjoint.

- We have

$$\begin{aligned}
\langle u, Av \rangle &= -i \left[\bar{u}(2\pi)v(2\pi) - \bar{u}(0)v(0)\right] + \langle Au, v \rangle \\
&= -i \left[\bar{u}(2\pi) - \bar{u}(0)\right] v(2\pi) + i\bar{u}(0) \left[v(2\pi) - v(0)\right] + \langle Au, v \rangle.
\end{aligned}$$

- If $u(2\pi) = u(0)$, then the boundary terms vanish if and only if $v(2\pi) = v(0)$, and then $\langle u, Av \rangle = \langle Au, v \rangle$. It follows that A with periodic boundary conditions,

$$A : D(A) \subset L^2(0, 2\pi) \to L^2(0, 2\pi),$$
$$D(A) = \left\{ u \in H^1(0, 2\pi) : u(0) = u(2\pi) \right\},$$

  is self-adjoint.

- The eigenfunctions u satisfy $-iu' = \lambda u$, so $u(x) = ce^{i\lambda x}$, and for nonzero solutions the periodic boundary condition $u(2\pi) = u(0)$ implies that $e^{2\pi i \lambda} = 1$, or $\lambda = n \in \mathbb{Z}$. Thus the eigenvalues and eigenfunctions are

$$\lambda_n = n, \qquad u_n(x) = e^{inx}, \qquad n \in \mathbb{Z}.$$

- The corresponding eigenfunction expansion of 2π-periodic functions is the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx.$$

- If $u(0) = 0$, then

$$\langle u, Av \rangle = -i\bar{u}(2\pi)v(2\pi) + \langle Au, v \rangle,$$

  so the adjoint boundary condition that ensures the boundary term vanishes is $v(2\pi) = 0$.

- Thus if

$$A : D(A) \subset L^2(0, 2\pi) \to L^2(0, 2\pi), \qquad A = -i\frac{d}{dx}$$
$$D(A) = \left\{ u \in H^1(0, 2\pi) : u(0) = 0 \right\},$$

  then

$$A^* : D(A^*) \subset L^2(0, 2\pi) \to L^2(0, 2\pi), \qquad A^* = -i\frac{d}{dx}$$
$$D(A^*) = \left\{ u \in H^1(0, 2\pi) : u(2\pi) = 0 \right\},$$

  and A is not self-adjoint since its boundary condition is not self-adjoint.

- (c) The general solution of the ODE is $u(x) = ce^{i\lambda x}$. The boundary condition $u(0) = 0$ implies that $c = 0$, so $u = 0$, and the operator A has no eigenvalues. (In fact, $A - \lambda I$ is invertible for every $\lambda \in \mathbb{C}$, so not only does A have no eigenvalues, but the spectrum of A is empty.)
- This behavior contrasts with that of linear maps on finite-dimensional vector spaces, which always have at least one eigenvalue and eigenfunction, even if they are not Hermitian.

3. Let A be a regular Sturm-Liouville operator, given by

$$Au = -(pu')' + qu,$$

acting in $L^2(a,b)$. Verify that A with the Robin boundary conditions

$$\alpha u'(a) + u(a) = 0, \qquad u'(b) + \beta u(b) = 0$$

is self-adjoint.

Solution.

- Using a real inner-product and integrating by parts, we get that

$$\begin{aligned}\langle Au, v\rangle - \langle u, Av\rangle &= \int_a^b \left[-(pu')' + qu\right] v - u\left[-(pv')' + qv\right]\,dx \\ &= \int_a^b (puv' - pu'v)'\,dx \\ &= \left[p(uv' - u'v)\right]_a^b.\end{aligned}$$

- If $\alpha u'(a) + u(a) = 0$, then

$$\begin{aligned}&p(a)\left[u(a)v'(a) - u'(a)v(a)\right] \\ &\quad = p(a)\left[\alpha u'(a) + u(a)\right]v'(a) - p(a)u'(a)\left[\alpha v'(a) + v(a)\right] \\ &\quad = -p(a)u(a)\left[\alpha v'(a) + v(a)\right],\end{aligned}$$

  so (provided that $p(a) \neq 0$, which is the case for a regular Sturm-Liouville operator) the boundary term at $x = a$ vanishes if and only if $\alpha v'(a) + v(a) = 0$, meaning that the BC is self-adjoint.

- Similarly, if $u'(b) + \beta u(b) = 0$, then

$$\begin{aligned}&p(b)\left[u(b)v'(b) - u'(b)v(b)\right] \\ &\quad = p(b)\left[u'(b) + \beta u(b)\right]v'(b) - p(b)u'(b)\left[v'(b) + \beta v(b)\right] \\ &\quad = -p(b)u(b)\left[v'(b) + \beta v(b)\right],\end{aligned}$$

  so the boundary term at $x = b$ vanishes if and only if $v'(b) + \beta v(b) = 0$.

4. Show that the eigenvalues of the Sturm-Liouville problem

$$-u'' = \lambda u \qquad 0 < x < 1$$
$$u(0) = 0, \qquad u'(1) + \beta u(1) = 0$$

are given by $\lambda = k^2$ where $k > 0$ satisfies the equation

$$\beta \tan k + k = 0.$$

Show graphically that there is a infinite sequence of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$ with $\lambda_n \to \infty$ as $n \to \infty$. What is the asymptotic behavior of λ_n as $n \to \infty$?

Solution.

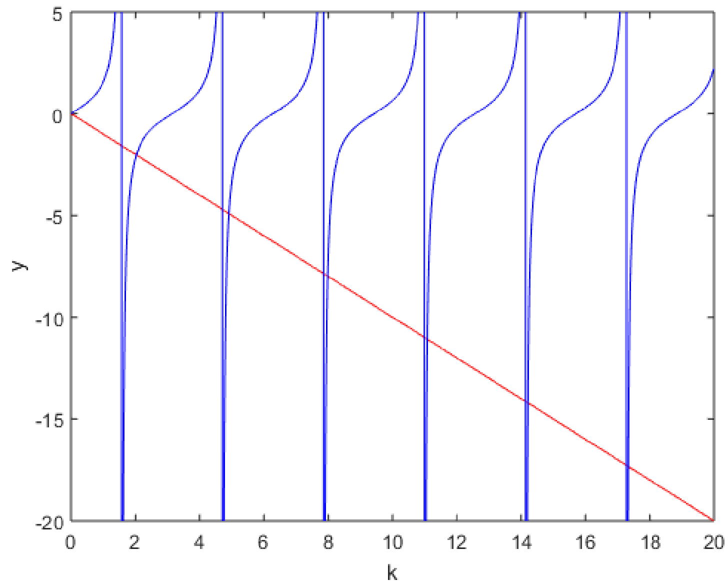
- From Problem 3, the eigenvalue problem is self-adjoint.
- If λ is an eigenvalue and u is a nonzero eigenfunction, then the self-adjointness implies that

$$\lambda \int_0^1 u^2\,dx = -\int u'' u\,dx = \int_0^1 (u')^2\,dx > 0,$$

  since $u' \neq 0$ for any nonzero solution of the BVP, so $\lambda = k^2 > 0$. The solution of the ODE with $u(0) = 0$ is then $u(x) = c \sin kx$, and the BC at $x = 1$ implies that $k \cos k + \beta \sin k = 0$, or $\beta \tan k + k = 0$.

- The roots of $\beta \tan k + k = 0$ are the k-coordinates of the intersections of the curve $y = -k/\beta$ with the curve $y = \tan k$, where we assume for definiteness that $\beta > 0$ (see figure below for $\beta = 1$). There are infinitely many such points and $k_n \sim (2n-1)\pi/2$, or

$$\lambda_n \sim \frac{(2n-1)^2 \pi^2}{4} \qquad \text{as } n \to \infty$$



5
0
-5
-10
-15
-20
y
0
2
4
6
8
10
12
14
16
18
20
k

5. The following IBVP describes heat flow in a rod whose left end is held at temperature 0 and whose right end loses heat to the surroundings according to Newton's law of cooling (heat flux is proportional to the temperature difference):

$$\begin{aligned} &u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \qquad u_x(1,t) = -\beta u(1,t) \\ &u(x,0) = f(x) \end{aligned}$$

Solve this IBVP by the method of separation of variables.

Solution.

- The separated solutions of the PDE and the BCs are

$$u(x,t) = e^{-\lambda_n t} \sin(k_n x)$$

where $\lambda_n = k_n^2$ are the eigenvalues from Problem 4.

- Superposing the separated solutions, we get that

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin(k_n x)$$

where the coefficients b_n are chosen so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(k_n x)$$

- Since eigenvalue problem is self-adjoint, the eigenfunctions are orthogonal, and

$$b_n = \frac{\int_0^1 f(x) \sin(k_n x)\, dx}{\int_0^1 \sin^2(k_n x)\, dx}.$$

- To verify the orthogonality explicitly, we use the addition formula for cosines, to get for $k_m \neq k_n$ that

$$\begin{aligned}
\int_0^1 \sin(k_m x)\sin(k_n x)\,dx &= \frac{1}{2}\left[\frac{\sin(k_m - k_n)}{k_m - k_n} - \frac{\sin(k_m + k_n)}{k_m + k_n}\right] \\
&= \frac{k_n \cos k_n \sin k_m - k_m \cos k_m \sin k_n}{k_m^2 - k_n^2} \\
&= \frac{\sin k_n \sin k_m - \sin k_m \sin k_n}{k_m^2 - k_n^2} \\
&= 0,
\end{aligned}$$

and, by a similar calculation,

$$\int_0^1 \sin^2(k_n x)\,dx = \frac{\beta + \cos^2 k_n}{2\beta}.$$