PROBLEM SET 5
Math 207B, Winter 2016
Due: Fri, Feb. 19

1. Suppose that $p : [a, b] \to \mathbb{R}$ is a continuously differentiable function such that $p > 0$, and $q, r : [a, b] \to \mathbb{R}$ are continuous functions such that $r > 0$, $q \geq 0$. Define a weighted inner product on $L^2(a, b)$ by

$$\langle u, v\rangle_r = \int_a^b r(x)\overline{u(x)}v(x)\, dx.$$

Let $A : D(A) \subset L^2(a, b) \to L^2(a, b)$ be the operator

$$A = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right]$$

with Dirichlet boundary conditions and domain

$$D(A) = \left\{u \in H^2(a, b) : u(a) = 0, u(b) = 0\right\}.$$

(a) Show that

$$\langle u, Av\rangle_r = \langle Au, v\rangle_r \qquad \text{for all } u, v \in D(A),$$

meaning that A is self-adjoint with respect to $\langle\cdot,\cdot\rangle_r$.
(b) Show that the eigenvalues λ of the weighted Sturm-Liouville eigenvalue problem

$$-(pu')' + qu = \lambda ru, \qquad u(a) = 0, \quad u(b) = 0$$

are real and positive and eigenfunctions associated with different eigenvalues are orthogonal with respect to $\langle\cdot,\cdot\rangle_r$.

2. A nonuniform string of length one with wave speed $c_0(x) = \sqrt{T/\rho_0(x)} > 0$ is fixed at each end, with zero initial displacement and nonzero initial velocity. The transverse displacement $y = u(x, t)$ of the string satisfies the IBVP

$$\begin{aligned}
&u_{tt} = c_0^2(x)u_{xx} \qquad 0 < x < 1, \quad t > 0,\\
&u(0, t) = 0, \qquad u(1, t) = 0 \qquad t > 0,\\
&u(x, 0) = 0 \qquad 0 < x < 1,\\
&u_t(x, 0) = g(x) \qquad 0 < x < 1,
\end{aligned}$$

Find the solution in terms of the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the weighted Sturm-Liouville problem

$$-c_0^2\phi_n'' = \lambda_n \phi_n, \qquad \phi_n(0) = 0, \quad \phi_n(1) = 0, \qquad n = 1, 2, 3, \ldots$$

3. The Fourier solution of the initial value problem

$$\begin{aligned}
&u_{tt} = u_{xx} \qquad 0 < x < 1, \quad t > 0,\\
&u(0,t) = 0, \qquad u(1,t) = 0 \qquad t > 0,\\
&u(x,0) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2(1-x) & \text{if } 1/2 < x < 1, \end{cases}\\
&u_t(x,0) = 0 \qquad 0 \le x \le 1,
\end{aligned}$$

is given by

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right]$$

(a) Show that the Fourier series converges to a continuous function. What order of spatial (weak) L^2-derivatives does $u(x,t)$ have?

(b) Verify from the Fourier solution that

$$\int_0^1 \left[u_t^2(x,t) + u_x^2(x,t)\right]\, dx = \text{constant} \qquad \text{for } -\infty < t < \infty.$$

(c) Use MATLAB (or another program) to compute the partial sum

$$u_N(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right]$$

at $t = 0.25$ for $N = 5$ and $N = 50$.

(d) Use the addition formula for sines to shows that the Fourier solution can be written in the form of the d'Alembert solution as

$$u(x,t) = F(x-t) + F(x+t)$$

for a suitable function $F : \mathbb{R} \to \mathbb{R}$. What is F?

4. Suppose that $u(x,t)$ is a smooth solution of the wave equation

$$u_{tt} = c_0^2 \Delta u,$$

where $x \in \mathbb{R}^n$, the wave speed $c_0 > 0$ is a constant.
(a) Show that u satisfies the energy equation

$$\frac{1}{2}\left(u_t^2 + c_0^2 |\nabla u|^2\right)_t - \nabla \cdot \left(c_0^2 u_t \nabla u\right) = 0.$$

(b) For $T > 0$, let $\Omega_T \subset \mathbb{R}^{n+1}$ be the space-time cone

$$\Omega_T = \left\{(x,t) \in \mathbb{R}^{n+1} : |x| < c_0(T-t),\, 0 < t < T\right\},$$

and for $0 \le t \le T$, let $B(T-t)$ be the spatial cross-section of Ω_T at time t

$$B(T-t) = \left\{x \in \mathbb{R}^n : |x| < c_0(T-t)\right\}.$$

Define

$$e_T(t) = \frac{1}{2}\int_{B(T-t)} \left(u_t^2 + c_0^2|\nabla u|^2\right)\, dx,$$

and show that $e_T(t) \le e_T(0)$.
(c) Suppose that u_1, u_2 are smooth solution of the wave equation such that

$$u_i(x,0) = f_i(x), \quad u_{it}(x,0) = g_i(x) \qquad i = 1,2$$

where $f_1 = f_2$, $g_1 = g_2$ in $|x| \le c_0 T$, show that $u_1 = u_2$ in Ω_T.
HINT. For (b), apply the divergence theorem in space-time to the equation in (a) over the truncated cone $\{(x,t') \in \Omega_T : 0 < t' < t\}$, and note that the space-time normal to the side of the cone Ω_T is $N = (\hat{x}, c_0)/\sqrt{1+c_0^2}$ where $\hat{x} = x/|x|$. For (c), consider $u = u_1 - u_2$.