PROBLEM SET 5: SOLUTIONS
Math 207B, Winter 2016

1. Suppose that $p : [a, b] \to \mathbb{R}$ is a continuously differentiable function such that $p > 0$, and $q, r : [a, b] \to \mathbb{R}$ are continuous functions such that $r > 0$, $q \geq 0$. Define a weighted inner product on $L^2(a, b)$ by

$$\langle u, v\rangle_r = \int_a^b r(x)\overline{u(x)}v(x)\, dx.$$

Let $A : D(A) \subset L^2(a, b) \to L^2(a, b)$ be the operator

$$A = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right]$$

with Dirichlet boundary conditions and domain

$$D(A) = \left\{u \in H^2(a, b) : u(a) = 0, u(b) = 0\right\}.$$

(a) Show that

$$\langle u, Av\rangle_r = \langle Au, v\rangle_r \qquad \text{for all } u, v \in D(A),$$

meaning that A is self-adjoint with respect to $\langle\cdot, \cdot\rangle_r$.
(b) Show that the eigenvalues λ of the weighted Sturm-Liouville eigenvalue problem

$$-(pu')' + qu = \lambda ru, \qquad u(a) = 0, \quad u(b) = 0$$

are real and positive and eigenfunctions associated with different eigenvalues are orthogonal with respect to $\langle\cdot, \cdot\rangle_r$.

Solution.

- (a) Using integration by parts and the boundary conditions satisfied by $u, v \in D(A)$, we have

$$\begin{aligned}
\langle u, Av\rangle_r &= \int_a^b \overline{u}\left[-(pv')' + qv\right]\, dx \\
&= \left[p(\bar{u}'v - \bar{u}v')\right]_a^b + \int_a^b \left[-(p\bar{u}')' + q\bar{u}\right] v\, dx \\
&= \int_a^b \overline{\left[-(pu')' + qu\right]}v\, dx \\
&= \langle Au, v\rangle_r.
\end{aligned}$$

- (b) The reality of the eigenvalues and the orthogonality of the eigenfunctions follows directly from the self-adjointness of A. Moreover, if $Au = \lambda u$ and $u \in D(A)$ is normalized so that $\|u\|_r = 1$, then an integration by parts gives

$$\lambda = \langle u, Au \rangle_r = \int_a^b \bar{u}[-(pu')' + qu]\, dx = \int_a^b \left[p|u'|^2 + q|u|^2\right]\, dx \geq 0.$$

- If $\lambda = 0$, then $\int_a^b p|u'|^2\, dx = 0$, so $u' = 0$ and $u =$ constant. Then the boundary condition implies that $u = 0$, so $\lambda = 0$ is not an eigenvalue, and $\lambda > 0$.

2. A nonuniform string of length one with wave speed $c_0(x) = \sqrt{T/\rho_0(x)} > 0$ is fixed at each end, with zero initial displacement and nonzero initial velocity. The transverse displacement $y = u(x,t)$ of the string satisfies the IBVP

$$
\begin{aligned}
&u_{tt} = c_0^2(x) u_{xx} \qquad 0 < x < 1, \quad t > 0,\\
&u(0,t) = 0, \qquad u(1,t) = 0 \qquad t > 0,\\
&u(x,0) = 0 \qquad 0 < x < 1,\\
&u_t(x,0) = g(x) \qquad 0 < x < 1,
\end{aligned}
$$

Find the solution in terms of the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the weighted Sturm-Liouville problem

$$-c_0^2 \phi_n'' = \lambda_n \phi_n, \qquad \phi_n(0) = 0, \quad \phi_n(1) = 0, \qquad n = 1, 2, 3, \dots$$

Solution.

- Separation of variables gives the solutions

$$u(x,t) = \begin{cases} \phi_n(x)\cos(k_n t), \\ \phi_n(x)\sin(k_n t), \end{cases}$$

  where $k_n > 0$ with $k_n^2 = \lambda_n$. From the previous question, $\lambda_n > 0$ and the eigenfuctions are orthogonal with respect to the weight $r = c_0^{-2}$.

- Superposing the separated solutions that are zero at $t = 0$, we get that

$$u(x,t) = \sum_{n=1}^{\infty} b_n \phi_n(x) \sin(k_n t).$$

- The initial condition for $u_t(x,0)$ is satisfied if $g(x) = \sum_{n=1}^{\infty} k_n b_n \phi_n(x)$. Using the orthogonality of the eigenfunctions, we get that

$$k_n b_n = \frac{\langle g, \phi_n \rangle_r}{\|\phi_n\|_r^2},$$

  or

$$b_n = \frac{\int_0^1 c_0^{-2} g \phi_n \, dx}{k_n \int_0^1 c_0^{-2} \phi_n^2 \, dx}.$$

3. The Fourier solution of the initial value problem

$$\begin{aligned} &u_{tt} = u_{xx} \qquad 0 < x < 1, \quad t > 0, \\ &u(0,t) = 0, \qquad u(1,t) = 0 \qquad t > 0, \\ &u(x,0) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2(1-x) & \text{if } 1/2 < x < 1, \end{cases} \\ &u_t(x,0) = 0 \qquad 0 \le x \le 1, \end{aligned}$$

is given by

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right]$$

(a) Show that the Fourier series converges to a continuous function. What order of spatial (weak) L^2-derivatives does $u(x,t)$ have?

(b) Verify from the Fourier solution that

$$\int_0^1 \left[u_t^2(x,t) + u_x^2(x,t)\right] \, dx = \text{constant} \qquad \text{for } -\infty < t < \infty.$$

(c) Use MATLAB (or another program) to compute the partial sum

$$u_N(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right]$$

at $t = 0.25$ for $N = 5$ and $N = 50$.

(d) Use the addition formula for sines to shows that the Fourier solution can be written in the form of the d'Alembert solution as

$$u(x,t) = F(x-t) + F(x+t)$$

for a suitable function $F : \mathbb{R} \to \mathbb{R}$. What is F?

Solution.

- (a) We have

$$\left| \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right] \right| \le \frac{1}{(2n-1)^2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} < \infty,$$

so the Weierstrass M-test implies that the series of continuous functions converges uniformly to a continuous function.

- We have

$$u_x(x,t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right],$$

so by Parseval's theorem (and the fact that $\|\cos m\pi x\|_{L^2}^2 = 1/2$)

$$\begin{aligned} \|u_x(\cdot,t)\|_{L^2}^2 &= \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos^2\left[(2n-1)\pi t\right] \\ &\le \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}, \end{aligned}$$

meaning that $u_x(\cdot,t) \in L^2(0,1)$ for all $t \in \mathbb{R}$.

- We have the distributional derivative

$$u_{xx}(x,t) = 8 \sum_{n=1}^{\infty} (-1)^n \sin\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right],$$

which does not belong to L^2 since $\sum_{n=1}^{\infty} \cos^2\left[(2n-1)\pi t\right]$ diverges (except for special values of t, such as $t = 1/2$). It follows that $u(\cdot,t) \in H^1(0,1)$ has one spatial L^2-derivative.

- More generally, if we also consider fractional derivatives of order s, then $u(\cdot,t) \in H^s(0,1)$ for $s < 3/2$.

- (b) We have

$$\begin{aligned} u_t(x,t) &= -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \sin\left[(2n-1)\pi x\right] \sin\left[(2n-1)\pi t\right], \\ u_x(x,t) &= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos\left[(2n-1)\pi x\right] \cos\left[(2n-1)\pi t\right]. \end{aligned}$$

Parseval's theorem implies that

$$\int_0^1 u_t^2(x,t)\,dx = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin^2\left[(2n-1)\pi t\right],$$

$$\int_0^1 u_x^2(x,t)\,dx = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos^2\left[(2n-1)\pi t\right],$$

so

$$\int_0^1 \left[u_t^2(x,t) + u_x^2(x,t)\right]\,dx = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

is a constant independent of time.

- At $t = 0$

  $$\int_0^1 \left[u_t^2(x,0) + u_x^2(x,0)\right]\,dx = \int_0^1 u_x^2(x,0)\,dx = 4,$$

  so we get the following sum from the Fourier expansion of the function $u_x(x,0)$:

  $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

- (c) The partial sums are shown below. Although the Fourier series converges uniformly, it isn't rapidly convergent since the solution isn't a smooth function of x.

- (d) Using the trigonometric identity

  $$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right],$$

  we get that

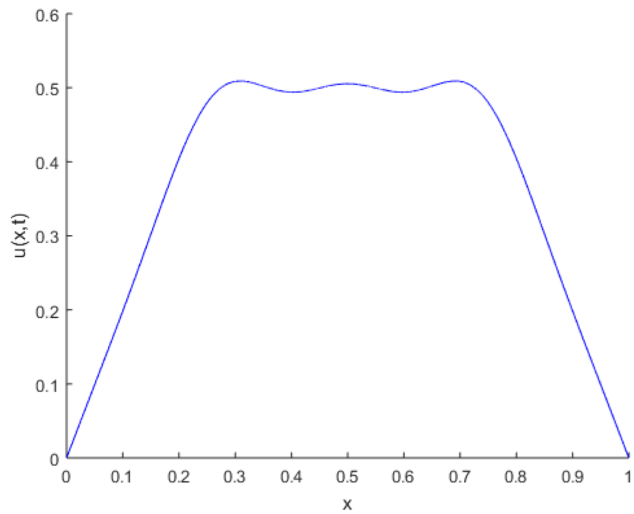
  $$\begin{aligned} u(x,t) &= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[(2n-1)\pi(x-t)\right] \\ &\quad + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[(2n-1)\pi(x+t)\right] \\ &= F(x-t) + F(x+t), \end{aligned}$$

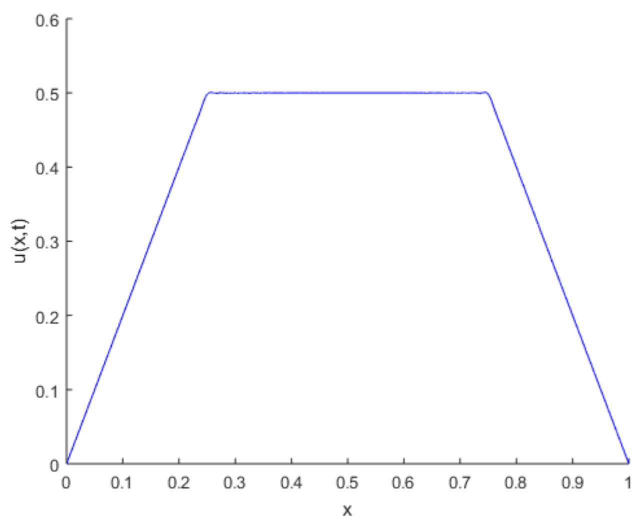
where

$$F(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[(2n-1)\pi x\right].$$

- The function $2F : \mathbb{R} \to \mathbb{R}$ is the odd, periodic extension of the initial data, meaning that $F(-x) = -F(x)$, $F(x+2) = F(x)$, and

$$F(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1/2 \\ 1-x & \text{if } 1/2 < x < 1, \end{cases}$$





4. Suppose that $u(x,t)$ is a smooth solution of the wave equation

$$u_{tt} = c_0^2 \Delta u,$$

where $x \in \mathbb{R}^n$, and the wave speed $c_0 > 0$ is a constant.
(a) Show that u satisfies the energy equation

$$\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)_t - \nabla \cdot \left(c_0^2 u_t \nabla u\right) = 0.$$

(b) For $T > 0$, let $\Omega_T \subset \mathbb{R}^{n+1}$ be the space-time cone

$$\Omega_T = \left\{(x,t) \in \mathbb{R}^{n+1} : |x| < c_0(T-t),\, 0 < t < T\right\},$$

and for $0 \le t \le T$, let $B(T-t)$ be the spatial cross-section of Ω_T at time t

$$B(T-t) = \{x \in \mathbb{R}^n : |x| < c_0(T-t)\}.$$

Define

$$e_T(t) = \frac{1}{2}\int_{B(T-t)} \left(u_t^2 + c_0^2|\nabla u|^2\right)\, dx,$$

and show that $e_T(t) \le e_T(0)$.
(c) Suppose that u_1, u_2 are smooth solution of the wave equation such that

$$u_i(x,0) = f_i(x), \quad u_{it}(x,0) = g_i(x) \qquad i = 1,2$$

where $f_1 = f_2$, $g_1 = g_2$ in $|x| \le c_0 T$, show that $u_1 = u_2$ in Ω_T.
HINT. For (b), apply the divergence theorem in space-time to the equation in (a) over the truncated cone $\{(x,t') \in \Omega_T : 0 < t' < t\}$, and note that the space-time normal to the side of the cone Ω_T is $N = (\hat{x}, c_0)/\sqrt{1+c_0^2}$ where $\hat{x} = x/|x|$. For (c), consider $u = u_1 - u_2$.

Solution.

- (a) Multiplying the wave equation by u_t, we have

  $$u_t u_{tt} - c_0^2 u_t \Delta u = 0.$$

  Using the identities

  $$u_t u_{tt} = \left(\frac{1}{2}u_t^2\right)_t, \qquad u_t \Delta u = \nabla \cdot (u_t \nabla u) - \left(\frac{1}{2}\nabla u \cdot \nabla u\right)_t,$$

  we get that

  $$\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)_t - \nabla \cdot \left(c_0^2 u_t \nabla u\right) = 0.$$

- For $0 < T' < T$, let $\Omega_{T',T}$ denote the truncated cone

$$\Omega_{T',T} = \left\{(x,t) \in \mathbb{R}^{n+1} : |x| < c_0(T-t) \text{ and } 0 < t < T'\right\}.$$

  The boundary of $\Omega_{T',T}$

$$\partial\Omega_{T',T} = \Gamma \cup \Sigma \cup \Gamma'$$

  consists of the bottom

$$\Gamma = \left\{(x,0) \in \mathbb{R}^{n+1} : |x| \leq c_0 T\right\}$$

  with outward space-time normal $N = (0,-1)$, the side

$$\Sigma = \left\{(x,t) \in \mathbb{R}^{n+1} : |x| = c_0(T-t) \text{ and } 0 < t < T'\right\}$$

  with outward space-time normal $N = (\hat{x}, c_0)/\sqrt{1+c_0^2}$, and the top

$$\Gamma' = \left\{(x,T') \in \mathbb{R}^{n+1} : |x| \leq c_0(T-T')\right\}$$

  with outward space-time normal $N = (0,1)$.

- Integrating the energy equation over $\Omega_{T',T}$ and applying the divergence theorem, we get that

$$\begin{aligned} 0 &= \int_{\Omega_{T',T}} \left\{\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)_t - \nabla \cdot \left(c_0^2 u_t \nabla u\right)\right\} dxdt \\ &= \int_{\partial\Omega_{T',T}} \left\{\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right)\nu - c_0^2 u_t \nabla u \cdot n\right\} dS \end{aligned}$$

  where $N = (n, \nu)$ is the outward space-time normal to $\partial\Omega_{T',T}$.

- Splitting the boundary integral into an integral over the bottom, side, and top, we get that

$$e_T(T') = e_T(0) - \frac{c_0}{\sqrt{1+c_0^2}} \int_\Sigma \left\{\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right) - c_0 u_t \nabla u \cdot \hat{x}\right\} dS,$$

  where

$$e_T(T') = \int_{\Gamma'} \frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right) dx, \quad e_T(0) = \int_\Gamma \frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right) dx.$$

- Completing the square, and using the fact that $\hat{x}$ is a unit vector, we get that

$$\frac{1}{2}\left(u_t^2 + c_0^2|\nabla u|^2\right) - c_0 u_t \nabla u \cdot \hat{x} = \frac{1}{2}\left(u_t\hat{x} - c_0\nabla u\right) \cdot \left(u_t\hat{x} - c_0\nabla u\right) \geq 0.$$

  It follows that $e_T(T') \leq e_T(0)$ for $0 < T' < T$.

- A physical interpretation of this inequality is that energy can only propagate out of the cone Ω_T, not into it.

- (c) If u_1, u_2 are two solutions with the same initial data for u_i and u_{it} in $|x| \leq c_0 T$, then $u = u_1 - u_2$ has zero initial data, in $|x| \leq c_0 T$ and therefore $e_T(0) = 0$. The previous inequality implies that $0 \leq e_T(t) \leq e_T(0)$ for $0 < t < T$, so $e_T(t) = 0$. It follows that $u_t = 0$ and $\nabla u = 0$ in Ω_T, meaning that $u =$ constant. Then the initial condition implies that $u = 0$ and $u_1 = u_2$ in Ω_T.

Remark. As this result shows, the solution of the wave equation at some point in space-time can only depend on, or influence, the solution at other points of space-time that can be reached by traveling at speeds less than or equal to c_0.