Problem set 6
Math 207B, Winter 2016
Due: Fri, Mar. 4

1. Suppose that $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are two solutions of the homogeneous Sturm-Liouville equation

$$-(pu')' + qu = 0$$

where $p, q : \mathbb{R} \to \mathbb{R}$ are smooth functions and $p > 0$. If $W = u_1 u_2' - u_2 u_1'$ is the Wronskian of u_1, u_2, show that $pW = \text{constant}$.

2. Compute the Green's function for the BVP

$$\begin{aligned} &-u'' + u = f(x) \qquad 0 < x < 1 \\ &u(0) = 0, \qquad u(1) = 0. \end{aligned}$$

Write down the integral representation of the solution u in terms of f.

3. Compute the Green's function for the BVP

$$\begin{aligned} &-u'' = f(x) \qquad 0 < x < 1 \\ &u(0) + u(1) = 0, \qquad u'(0) + u'(1) = 0. \end{aligned}$$

Write down the integral representation of the solution u in terms of f.

4. Compute the generalized Green's function $G(x, \xi)$ for the BVP

$$\begin{aligned} &-u'' = \pi^2 u + f(x) \qquad 0 < x < 1 \\ &u(0) = 0, \qquad u(1) = 0. \end{aligned}$$

State the equations that are satisfied by the function

$$u(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi.$$

5. Consider the Sturm-Liouville equation

$$-(pu')' + qu = \lambda r u, \qquad a < x < b$$

where $p, q, r : [a, b] \to \mathbb{R}$ are smooth functions and $p(x), r(x) > 0$ for $a \leq x \leq b$. Show that the change of variables

$$t = \int_a^x \sqrt{\frac{r(s)}{p(s)}}\, ds, \qquad v(t) = \left[r(x)p(x)\right]^{1/4} u(x)$$

transforms this equation into a Sturm-Liouville equation with $p = r = 1$ of the form

$$-v'' + Qv = \lambda v, \qquad 0 < t < c.$$

What are c and $Q : [0, c] \to \mathbb{R}$?