Problem set 6: Solutions
Math 207B, Winter 2016

1. Suppose that $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are two solutions of the homogeneous Sturm-Liouville equation

$$-(pu')' + qu = 0$$

where $p, q : \mathbb{R} \to \mathbb{R}$ are smooth functions and $p > 0$. If $W = u_1 u_2' - u_2 u_1'$ is the Wronskian of u_1, u_2, show that $pW =$ constant.

Solution.

- Using the ODE, we compute that

$$\begin{aligned}(pW)' &= (u_1 \cdot pu_2' - u_2 \cdot pu_1')' \\ &= u_1 \cdot (pu_2)' + u_1' \cdot pu_2' - u_2 \cdot (pu_1)' - u_2' \cdot pu_1' \\ &= u_1 \cdot qu_2 - u_2 \cdot qu_1 \\ &= 0\end{aligned}$$

  so pW is a constant.

2. Compute the Green's function for the BVP

$$-u'' + u = f(x) \qquad 0 < x < 1$$
$$u(0) = 0, \qquad u(1) = 0.$$

Write down the integral representation of the solution u in terms of f.

Solution.

- The Green's function $G(x, \xi)$ satisfies
$$-G_{xx} + G = \delta(x - \xi) \qquad 0 < x < 1$$
$$G(0, \xi) = 0, \qquad G(1, \xi) = 0.$$

- Solving the homogeneous ODE for $0 < x < \xi$ and $\xi < x < 1$ and imposing the appropriate boundary conditions, we get that
$$G(x, \xi) = \begin{cases} A(\xi)\sinh x & \text{if } 0 \le x < \xi \\ B(\xi)\sinh(1 - x) & \text{if } \xi < x \le 1 \end{cases}$$

- Imposition of the continuity of $G(x, \xi)$ at $x = \xi$ and the jump-condition
$$-G_x(\xi^+, \xi) + G_x(\xi^-, \xi) = 1,$$
gives the equations
$$A(\xi)\sinh\xi - B(\xi)\sinh(1 - \xi) = 0,$$
$$A(\xi)\cosh\xi + B(\xi)\cosh(1 - \xi) = 1.$$
Using the addition formula
$$\sinh\xi\cosh(1 - \xi) + \cosh\xi\sinh(1 - \xi) = \sinh 1,$$
we get the solution
$$A(\xi) = \frac{\sinh(1 - \xi)}{\sinh 1}, \qquad B(\xi) = \frac{\sinh\xi}{\sinh 1},$$
so the Green's function is
$$G(x, \xi) = \frac{\sinh(x_<)\sinh(1 - x_>)}{\sinh 1}$$
where $x_< = \min(x, \xi)$, $x_> = \max(x, \xi)$.

- The Green's function representation is
$$u(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi.$$

3. Compute the Green's function for the BVP

$$\begin{aligned} &-u'' = f(x) \qquad 0 < x < 1 \\ &u(0) + u(1) = 0, \qquad u'(0) + u'(1) = 0. \end{aligned}$$

Write down the integral representation of the solution u in terms of f.

Solution.

- The Green's function $G(x, \xi)$ satisfies

$$\begin{aligned} &-G_{xx} = \delta(x - \xi), \\ &G(0, \xi) + G(1, \xi) = 0, \qquad G_x(0, \xi) + G_x(1, \xi) = 0. \end{aligned}$$

- The boundary conditions are not separated, so we use general solutions of the homogeneous equation in $x < \xi$ and $x > \xi$ to get

$$G(x, \xi) = \begin{cases} A(\xi) + B(\xi)x & \text{if } 0 \leq x < \xi \\ C(\xi) + D(\xi)(1 - x) & \text{if } \xi < x \leq 1 \end{cases}$$

  The boundary conditions give

$$A + C = 0, \qquad B - D = 0.$$

  The continuity of G and the jump condition $-[G_x] = 1$ give

$$A + B\xi = C + D(1 - \xi), \qquad B + D = 1.$$

- It follows that $B = D = 1/2$ and

$$A = -C = \frac{1}{4} - \frac{1}{2}\xi,$$

  so

$$G(x, \xi) = \begin{cases} 1/4 + (x - \xi)/2 & \text{if } 0 \leq x < \xi \\ 1/4 + (\xi - x)/2 & \text{if } \xi < x \leq 1 \end{cases} = \frac{1}{4} - \frac{1}{2}|x - \xi|.$$

- The Green's function representation is

$$u(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi.$$

4. Compute the generalized Green's function $G(x, \xi)$ for the BVP

$$\begin{aligned} -u'' &= \pi^2 u + f(x) \qquad 0 < x < 1 \\ u(0) &= 0, \qquad u(1) = 0. \end{aligned}$$

State the equations that are satisfied by the function

$$u(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi.$$

Solution.

- A normalized solution of the homogeneous problem, with L^2-norm one, is $\phi(x) = \sqrt{2} \sin(\pi x)$.

- The generalized Green's function $G(x, \xi)$ satisfies

$$\begin{aligned} &-G_{xx} - \pi^2 G = \delta(x - \xi) - 2\sin(\pi\xi)\sin(\pi x) \qquad 0 < x < 1 \\ &G(0, \xi) = 0, \qquad G(1, \xi) = 0, \\ &\int_0^1 G(x, \xi) \sin(\pi x)\, dx = 0. \end{aligned}$$

  The right-hand side of the ODE is the projection of the δ-function onto the space orthogonal to ϕ to ensure that a solution exists, and the condition $G(\cdot, \xi) \perp \phi$ specifies a unique solution.

- In $x < \xi$ and $x > \xi$, we have

$$-G_{xx} - \pi^2 G = -2\sin(\pi\xi)\sin(\pi x).$$

- A particular solution of the non-homogeneous ODE

$$-u'' - \pi^2 u = C \sin(\pi x),$$

  whose right-hand side is a solution of the homogeneous ODE, is

$$u(x) = \frac{C}{2\pi} x \cos(\pi x),$$

  so the general solution is

$$u(x) = A\cos(\pi x) + B\sin(\pi x) + \frac{C}{2\pi} x \cos(\pi x).$$

- Imposing the appropriate boundary conditions on G, we get that

$$G(x, \xi) = A(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x) \quad \text{if } 0 \le x < \xi$$
$$G(x, \xi) = B(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x) + \frac{1}{\pi} \sin(\pi \xi) \cos(\pi x) \quad \text{if } \xi < x \le 1.$$

- The continuity of G at $x = \xi$ implies that

$$\pi(A - B) = \cos(\pi \xi).$$

One can verify directly that the jump condition $-[G_x] = 1$ at $x = \xi$ gives the same equation, so it is also satisfied, and therefore

$$G(x, \xi) = B(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x) + \frac{1}{\pi} \cos(\pi \xi) \sin(\pi x) \quad \text{if } 0 \le x < \xi$$
$$G(x, \xi) = B(\xi) \sin(\pi x) - \frac{1}{\pi} \sin(\pi \xi) x \cos(\pi x) + \frac{1}{\pi} \sin(\pi \xi) \cos(\pi x) \quad \text{if } \xi < x \le 1.$$

- The orthogonality condition $G(\cdot, \xi) \perp \phi$ gives, after some algebra,

$$B = -\frac{1}{\pi} \xi \cos(\pi \xi).$$

- The generalized Green's function can then be written as

$$G(x, \xi) = \frac{1}{\pi} \cos(\pi x_>) \sin(\pi x_<) - \frac{1}{\pi} x_> \cos(\pi x_>) \sin(\pi x_<) - \frac{1}{\pi} x_< \cos(\pi x_<) \sin(\pi x_>)$$

where $x_< = \min(x, \xi)$, $x_> = \max(x, \xi)$.

- The function $u(x)$ satisfies

$$-u'' = \pi^2 u + f(x) - 2 \left(\int_0^1 f(\xi) \sin(\pi \xi)\, d\xi \right) \sin(\pi x),$$
$$u(0) = 0, \qquad u(1) = 0,$$
$$\int_0^1 u(x) \sin(\pi x)\, dx = 0.$$

5. Consider the Sturm-Liouville equation

$$-(pu')' + qu = \lambda r u, \qquad a < x < b$$

where $p, q, r : [a, b] \to \mathbb{R}$ are smooth functions and $p(x), r(x) > 0$ for $a \leq x \leq b$. Show that the change of variables

$$t = \int_a^x \sqrt{\frac{r(s)}{p(s)}}\, ds, \qquad v(t) = [r(x)p(x)]^{1/4}\, u(x)$$

transforms this equation into a Sturm-Liouville equation with $p = r = 1$ of the form

$$-v'' + Qv = \lambda v, \qquad 0 < t < c.$$

What are c and $Q : [0, c] \to \mathbb{R}$?

Solution.

- This is an exercise in the chain rule. One finds that

$$Q = q - \frac{(pr)^{1/4}}{r}\left[p\left(\frac{1}{(pr)^{1/4}}\right)'\right]', \qquad c = \int_a^b \sqrt{\frac{r(s)}{p(s)}}\, ds.$$

  This transformation is called the Liouville transformation, and it shows that every Sturm-Liouville equation can be transformed to a normal form with $p = r = 1$.