PROBLEM SET 7
Math 207B, Winter 2016
Due: Mon, Mar. 14

1. (a) Use separation of variables to find the eigenvalues λ and eigenfunctions $u(x, y)$ of the Dirichlet Laplacian on the unit square that satisfy

$$\begin{aligned} -\left(u_{xx} + u_{yy}\right) &= \lambda u \qquad 0 < x < 1,\ 0 < y < 1 \\ u(x,0) = 0, \quad u(x,1) &= 0 \qquad 0 \le x \le 1 \\ u(0,y) = 0, \quad u(1,y) &= 0 \qquad 0 \le y \le 1. \end{aligned}$$

(b) What is the smallest eigenvalue that is not a simple eigenvalue?

2. (a) Let $\vec{x} = (x, y)$, $\vec{\xi} = (\xi, \eta)$, and $\vec{\xi}^* = (\xi, -\eta)$ where $\eta > 0$. Show that

$$G(\vec{x}, \vec{\xi}) = -\frac{1}{2\pi} \log\left(\frac{|\vec{x} - \vec{\xi}|}{|\vec{x} - \vec{\xi}^*|}\right)$$

is the solution of

$$\begin{aligned} -\left(G_{xx} + G_{yy}\right) = \delta(\vec{x} - \vec{\xi}) & \qquad \text{in } -\infty < x < \infty,\ y > 0 \\ G(\vec{x}, \vec{\xi}) = 0 & \qquad \text{on } y = 0. \end{aligned}$$

(b) Write down the Green's function representation for the solution $u(x, y)$ of the Dirichlet problem for the Laplacian in the upper half plane

$$\begin{aligned} u_{xx} + u_{yy} = 0 & \qquad \text{in } -\infty < x < \infty,\ y > 0 \\ u(x,0) = f(x). & \end{aligned}$$

You can assume that $u(x, y) \to 0$ sufficiently rapidly as $|(x, y)| \to \infty$.
(c) Use the Green's function representation to show that

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2}\, dt.$$