PROBLEM SET 2
Math 218A, Winter 2001

1. Obtain the self-similar solution of

$$
\begin{aligned}
&u_t = u_{xx},\\
&u(x,0) = \begin{cases} 1 & \text{if } x < 0,\\ 0 & \text{if } x > 0. \end{cases}
\end{aligned}
$$

2. The porous medium equation in cylindrical coordinates is

$$u_t = \frac{1}{r}\left(r\left(u^2\right)_r\right)_r.$$

Look for similarity solutions of the form

$$u(x,t) = \frac{1}{t^\alpha}\phi\left(\frac{r}{t^\beta}\right).$$

Show that the choice $\alpha = 1/2$, $\beta = 1/4$ is consistent with the equation, and in that case derive a similarity solution which has compact support in r. (This solution, due to Barenblatt, describes the spread of a wetting front through the porous medium.)

3. Suppose that $f \in C_c(\mathbb{R}^n)$ is compactly supported, and

$$u \in C_1^2\left(\mathbb{R}^n \times (0,\infty)\right) \cap C\left(\mathbb{R}^n \times [0,\infty)\right)$$

is the bounded solution of

$$
\begin{aligned}
&u_t = \Delta u,\\
&u(x,0) = f(x).
\end{aligned}
$$

Prove that for $t > 0$,

$$
\begin{aligned}
&\|u(\cdot,t)\|_2 \le \|f\|_2,\\
&\|u(\cdot,t\|_\infty \le \frac{1}{(4\pi t)^{n/2}}\|f\|_1,
\end{aligned}
$$

where $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^n)$-norm.

4. Let Ω be a smooth bounded domain. Suppose that

$$\begin{aligned}
&u_t + \mathbf{v} \cdot \nabla u = \Delta u \qquad \text{in } \Omega, \\
&\frac{\partial u}{\partial n} + \alpha u \geq 0 \qquad \text{on } \partial\Omega, \\
&u(x,0) \geq 0 \qquad \text{in } \Omega
\end{aligned}$$

where $\mathbf{v}$ is a constant vector, and $\alpha \geq 0$ is a nonnegative, constant scalar. Assuming that the strong maximum principle holds for this equation, prove that $u \geq 0$ for $x \in \Omega$ and $t > 0$. Give a physical interpretation in terms of heat flow.