Problem set 3
Math 218A, Winter 2001

1. The Tricomi equation

$$yu_{xx} - u_{yy} = 0$$

is a fundamental example of an equation of mixed type, being hyperbolic in $y > 0$ and elliptic in $y < 0$. It arises in the modeling of transonic fluid flows. Show that the change of variables

$$\eta = \frac{2}{3} y^{3/2}$$

in $y > 0$ transforms the Tricomi equation to an Euler-Poisson-Darboux equation

$$u_{xx} - u_{\eta\eta} - \frac{1}{3} u_\eta = 0.$$

2. Does the maximum principle hold for the wave equation?

3. Show that the solution of the nonhomogeneous wave equation,

$$\begin{aligned} u_{tt} - \Delta u &= f(x,t), \\ u(x,0) &= u_t(x,0) = 0, \end{aligned}$$

in three space dimensions is given by

$$u(x,t) = \frac{1}{4\pi} \int_{B_t(x)} \frac{f(y, t - |x-y|)}{|x-y|} \, dy.$$

Give a physical interpretion of this result.

4. Let S be the following action functional

$$S[u] = \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^n} \left(u_t^2 - |\nabla u|^2 \right) \, dx dt.$$

Show that if u is stationary point of S, meaning that

$$\frac{d}{d\epsilon} S[u + \epsilon \phi] \big|_{\epsilon = 0} = 0$$

for all $\phi \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$, then u satisfies the wave equation. You can assume, for simplicity, that $u \in C_c^\infty(\mathbb{R}^n \times [0,T])$ for each $T > 0$. Does u minimize S?

Note that the action is the time integral of the Lagrangian

$$L[u] = \frac{1}{2}\int_{\mathbb{R}^n} \left(u_t^2 - |\nabla u|^2\right)\, dx,$$

which is the *difference* between the kinetic and potential energies, whereas the conserved Hamiltonian

$$H[u] = \frac{1}{2}\int_{\mathbb{R}^n} \left(u_t^2 + |\nabla u|^2\right)\, dx$$

is the *sum* of the kinetic and potential energies.