PROBLEM SET 5

Math 218A, Winter 2001

1. Use Fourier transforms to solve the initial value problem for the wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0 \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

Show that for $f, g \in \mathcal{S}(\mathbb{R}^n)$ there is a unique solution $u \in C^\infty(\mathbb{R}; \mathcal{S})$. Show that

$$\|u_t\|_{H^s}^2 + \|\nabla u\|_{H^{s-1}}^2$$

is constant. Let D^s be the closure of $\mathcal{S}$ with respect to the norm

$$\|u\|_{D^s} = \|\nabla u\|_{H^{s-1}}^2.$$

Deduce that there exists a unique generalized solution (defined as a limit of Schwartz-valued solutions)

$$u \in C(\mathbb{R}; D^s) \cap C^1(\mathbb{R}; H^{s-1})$$

for each $f \in D^s$, $g \in H^{s-1}$.

2. Which of the following PDEs leads to a well-posed initial value problem, with data given at $t = 0$?

$$\begin{aligned} u_{tt} - u_{xx} &= u, \\ u_{tt} &= u_x, \\ u_{tt} &= u_{xx} - u_{yy}. \end{aligned}$$

Remarks. The second equation is the “sideways heat equation;” with x and t exchanged, the initial value problem corresponds to the determination of the temperature inside a rod given the temperature and heat flux at one end of the rod. The third equation is the wave equation with x and t exchanged; the initial value problem corresponds to the imposition of Cauchy data on a spacelike initial surface.

3. Use Fourier transforms to solve the system

$$
\begin{aligned}
&u_t + v_x = 0, \\
&v_t = 0, \\
&u(x,0) = f(x), \\
&v(x,0) = g(x),
\end{aligned}
$$

for $f, g \in \mathcal{S}(\mathbb{R})$. Show that the system satisfies the Hadamard-Petrowsky condition. Show that the solution satisfies the estimate

$$
\|(u,v)(t)|_{H^s} \le C\|(u,v)(0)|_{H^{s+1}}.
$$

(Note the "loss" of one derivative in this estimate; as a result, the system is said to be weakly well-posed.) Show that the system can be made ill-posed by the introduction of a linear constant coefficient lower order term.

4. Solve the heat equation

$$
\begin{aligned}
&u_t = u_{xx}, \\
&u(x,0) = f(x),
\end{aligned}
$$

by use of Laplace transforms in time.