# Problem set 6

Math 218A, Winter 2001

**1.** Use the method of characteristics to solve the eikonal equation

$$u_x^2 + u_y^2 = 1$$

with initial data given parametrically on the curve $y^2 = 2x + 1$ by $x = x_0(s)$, $y = y_0(s)$, $u = u_0(s)$, where

$$x_0(s) = \frac{\cos s}{1 - \cos s}, \quad y_0(s) = \frac{\sin s}{1 - \cos s}, \quad u_0(s) = \frac{\cos s}{1 - \cos s}.$$

Show that you get two solutions, depending on the choice of compatible initial data for $p = u_x$ and $q = u_y$, and that they are given by

$$u = x, \qquad u = \sqrt{x^2 + y^2} - 1.$$

**2.** Compute explicitly the Hopf-Lax solution of the Hamilton-Jacobi equation

$$\phi_t + \frac{1}{2}\phi_x^2 = 0,$$

with the following two sets of initial data:

$$\phi(x,0) = \begin{cases} x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases}$$

$$\phi(x,0) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0, \end{cases}$$

Sketch the projections of the characteristics onto $(x,t)$-space. Compare your result with the corresponding entropy solution of the inviscid Burgers equation for $u = \phi_x$.

**3.** Problems 6 and 7 in §3.5, p. 163 of Evans.

**4.** Use the method of characteristics and shock-fitting to find an entropy solution of the initial value problem

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0,$$

$$u(x,1) = \begin{cases} x & \text{if } -p < x < q, \\ 0 & \text{otherwise,} \end{cases}$$

where $p, q \geq 0$. At what rate does $\|u\|_\infty$ decay as $t \to +\infty$?

REMARK. This solution is called an "N-wave". It is the time-asymptotic state of any compactly supported solution of the inviscid Burger's equation. The front and rear shocks in an N-wave can be heard in sonic booms.