CHAPTER 1

Preliminaries

In this chapter, we collect various definitions and theorems for future use. Proofs may be found in the references e.g. [**7, 13, 16, 17, 18**].

1.1. Euclidean space

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. We denote the Euclidean norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

and the inner product of vectors $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We denote Lebesgue measure on $\mathbb{R}^n$ by dx, and the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$.

If E is a subset of $\mathbb{R}^n$, we denote the complement by $E^c = \mathbb{R}^n \setminus E$, the closure by $\overline{E}$, the interior by E° and the boundary by $\partial E = \overline{E} \setminus E^\circ$. The characteristic function $\chi_E : \mathbb{R}^n \to \mathbb{R}$ of E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

A set E is bounded if $\{|x| : x \in E\}$ is bounded in $\mathbb{R}$. A set is connected if it is not the disjoint union of two nonempty relatively open subsets. We sometimes refer to a connected open set as a domain.

We say that an open set Ω' in $\mathbb{R}^n$ is compactly contained in an open set Ω, written $\Omega' \Subset \Omega$, if $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact. If $\overline{\Omega'} \subset \Omega$, then

$$\operatorname{dist}(\Omega', \partial\Omega) = \inf\{|x - y| : x \in \Omega', y \in \partial\Omega\} > 0.$$

This distance is finite provided that $\Omega' \neq \emptyset$ and $\Omega \neq \mathbb{R}^n$.

1.2. Spaces of continuous functions

Let Ω be an open set in $\mathbb{R}^n$. We denote the space of continuous functions $u : \Omega \to \mathbb{R}$ by $C(\Omega)$; the space of functions with continuous partial derivatives in Ω of order less than or equal to $k \in \mathbb{N}$ by $C^k(\Omega)$; and the space of functions with continuous derivatives of all orders by $C^\infty(\Omega)$. Functions in these spaces need not be bounded even if Ω is bounded; for example, $(1/x) \in C^\infty(0, 1)$.

If Ω is a bounded open set in $\mathbb{R}^n$, we denote by $C(\overline{\Omega})$ the space of continuous functions $u : \overline{\Omega} \to \mathbb{R}$. This is a Banach space with respect to the maximum, or supremum, norm

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|.$$

We denote the support of a continuous function $u : \Omega \to \mathbb{R}^n$ by

$$\operatorname{spt} u = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We denote by $C_c(\Omega)$ the space of continuous functions whose support is compactly contained in Ω, and by $C_c^\infty(\Omega)$ the space of functions with continuous derivatives of all orders and compact support in Ω. We will sometimes refer to such functions as test functions.

The completion of $C_c(\mathbb{R}^n)$ with respect to the uniform norm is the space $C_0(\mathbb{R}^n)$ of continuous functions that approach zero at infinity. (Note that in many places the notation C_0 and C_0^∞ is used to denote the spaces of compactly supported functions that we denote by C_c and C_c^∞.)

If Ω is bounded, we say that a function $u : \overline{\Omega} \to \mathbb{R}$ belongs to $C^k(\overline{\Omega})$ if it is continuous and its partial derivatives of order less than or equal to k are uniformly continuous in Ω, in which case they extend to continuous functions on $\overline{\Omega}$. The space $C^k(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq k} \sup_{\Omega} |\partial^\alpha u|$$

where we use the multi-index notation for partial derivatives explained in Section 1.8. This norm is finite because the derivatives $\partial^\alpha u$ are continuous functions on the compact set $\overline{\Omega}$.

A vector field $X : \Omega \to \mathbb{R}^m$ belongs to $C^k(\overline{\Omega})$ if each of its components belongs to $C^k(\overline{\Omega})$.

1.3. Hölder spaces

The definition of continuity is not a quantitative one, because it does not say how rapidly the values $u(y)$ of a function approach its value $u(x)$ as $y \to x$. The modulus of continuity $\omega : [0, \infty] \to [0, \infty]$ of a general continuous function u, satisfying

$$|u(x) - u(y)| \leq \omega\left(|x - y|\right),$$

may decrease arbitrarily slowly. As a result, despite their simple and natural appearance, spaces of continuous functions are often not suitable for the analysis of PDEs, which is almost always based on quantitative estimates.

A straightforward and useful way to strengthen the definition of continuity is to require that the modulus of continuity is proportional to a power $|x - y|^\alpha$ for some exponent $0 < \alpha \leq 1$. Such functions are said to be Hölder continuous, or Lipschitz continuous if $\alpha = 1$. Roughly speaking, one can think of Hölder continuous functions with exponent α as functions with bounded fractional derivatives of the the order α.

Definition 1.1. Suppose that Ω is an open set in $\mathbb{R}^n$ and $0 < \alpha \leq 1$. A function $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent α in Ω if the quantity

$$[u]_{\alpha,\Omega} = \sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \tag{1.1}$$

is finite. A function $u : \Omega \to \mathbb{R}$ is locally uniformly Hölder continuous with exponent α in Ω if $[u]_{\alpha,\Omega'}$ is finite for every $\Omega' \Subset \Omega$. We denote by $C^{0,\alpha}(\Omega)$ the space of locally uniformly Hölder continuous functions with exponent α in Ω. If Ω is bounded, we denote by $C^{0,\alpha}\left(\overline{\Omega}\right)$ the space of uniformly Hölder continuous functions with exponent α in Ω.

We typically use Greek letters such as α, β both for Hölder exponents and multi-indices; it should be clear from the context which they denote.

When α and Ω are understood, we will abbreviate 'u is (locally) uniformly Hölder continuous with exponent α in Ω' to 'u is (locally) Hölder continuous.' If u is Hölder continuous with exponent one, then we say that u is Lipschitz continuous. There is no purpose in considering Hölder continuous functions with exponent greater than one, since any such function is differentiable with zero derivative, and is therefore constant.

The quantity $[u]_{\alpha,\Omega}$ is a semi-norm, but it is not a norm since it is zero for constant functions. The space $C^{0,\alpha}\left(\overline{\Omega}\right)$, where Ω is bounded, is a Banach space with respect to the norm

$$\|u\|_{C^{0,\alpha}\left(\overline{\Omega}\right)} = \sup_{\Omega} |u| + [u]_{\alpha,\Omega}\,.$$

EXAMPLE 1.2. For $0 < \alpha < 1$, define $u(x) : (0,1) \to \mathbb{R}$ by $u(x) = |x|^\alpha$. Then $u \in C^{0,\alpha}\left([0,1]\right)$, but $u \notin C^{0,\beta}\left([0,1]\right)$ for $\alpha < \beta \leq 1$.

EXAMPLE 1.3. The function $u(x) : (-1,1) \to \mathbb{R}$ given by $u(x) = |x|$ is Lipschitz continuous, but not continuously differentiable. Thus, $u \in C^{0,1}\left([-1,1]\right)$, but $u \notin C^{1}\left([-1,1]\right)$.

We may also define spaces of continuously differentiable functions whose kth derivative is Hölder continuous.

DEFINITION 1.4. If Ω is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $0 < \alpha \leq 1$, then $C^{k,\alpha}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$ with continuous partial derivatives in Ω of order less than or equal to k whose kth partial derivatives are locally uniformly Hölder continuous with exponent α in Ω. If the open set Ω is bounded, then $C^{k,\alpha}\left(\overline{\Omega}\right)$ consists of functions with uniformly continuous partial derivatives in Ω of order less than or equal to k whose kth partial derivatives are uniformly Hölder continuous with exponent α in Ω.

The space $C^{k,\alpha}\left(\overline{\Omega}\right)$ is a Banach space with respect to the norm

$$\|u\|_{C^{k,\alpha}\left(\overline{\Omega}\right)} = \sum_{|\beta|\leq k} \sup_{\Omega} \left|\partial^\beta u\right| + \sum_{|\beta|=k} \left[\partial^\beta u\right]_{\alpha,\Omega}$$

1.4. L^p spaces

As before, let Ω be an open set in $\mathbb{R}^n$ (or, more generally, a Lebesgue-measurable set).

DEFINITION 1.5. For $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of the Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f|^p \, dx < \infty,$$

and $L^\infty(\Omega)$ consists of the essentially bounded functions.

These spaces are Banach spaces with respect to the norms

$$\|f\|_p = \left(\int_\Omega |f|^p \, dx\right)^{1/p}, \qquad \|f\|_\infty = \sup_\Omega |f|$$

where sup denotes the essential supremum,

$$\sup_\Omega f = \inf \{M \in \mathbb{R} : f \leq M \text{ almost everywhere in } \Omega\}.$$

Strictly speaking, elements of the Banach space L^p are equivalence classes of functions that are equal almost everywhere, but we identify a function with its equivalence class unless we need to refer to the pointwise values of a specific representative. For example, we say that a function $f \in L^p(\Omega)$ is continuous if it is equal almost everywhere to a continuous function, and that it has compact support if it is equal almost everywhere to a function with compact support.

Next we summarize some fundamental inequalities for integrals, in addition to Minkowski's inequality which is implicit in the statement that $\|\cdot\|_{L^p}$ is a norm for $p \geq 1$.

Jensen's inequality states that the value of a convex function at a mean is less than or equal to the mean of the values of the convex function.

Theorem 1.6. *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, Ω is a set in $\mathbb{R}^n$ with finite Lebesgue measure, and $f \in L^1(\Omega)$. Then*

$$\phi\left(\frac{1}{|\Omega|}\int_\Omega f \, dx\right) \leq \frac{1}{|\Omega|}\int_\Omega \phi \circ f \, dx.$$

To state the next inequality, we first define the Hölder conjugate of an exponent p. We denote it by p' to distinguish it from the Sobolev conjugate p^* which we will introduce later on.

Definition 1.7. The Hölder conjugate of $p \in [1, \infty]$ is the quantity $p' \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the convention that $1/\infty = 0$.

The following result is called Hölder's inequality.[1] The special case when $p = p' = 1/2$ is the Cauchy-Schwartz inequality.

Theorem 1.8. *If $1 \leq p \leq \infty$, $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

Repeated application of this inequality gives the following generalization.

Theorem 1.9. *If $1 \leq p_i \leq \infty$ for $1 \leq i \leq N$ satisfy*

$$\sum_{i=1}^N \frac{1}{p_i} = 1$$

[1] In retrospect, it would've been better to use $L^{1/p}$ spaces instead of L^p spaces, just as it would've been better to use inverse temperature instead of temperature, with absolute zero corresponding to infinite coldness.

and $f_i \in L^{p_i}(\Omega)$ *for* $1 \le i \le N$*, then* $f = \prod_{i=1}^N f_i \in L^1(\Omega)$ *and*

$$\|f\|_1 \le \prod_{i=1}^N \|f_i\|_{p_i}.$$

If Ω has finite measure and $1 \le q \le p$, then Hölder's inequality shows that $f \in L^p(\Omega)$ implies that $f \in L^q(\Omega)$ and

$$\|f\|_q \le |\Omega|^{1/q-1/p}\|f\|_p.$$

Thus, the embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. This result is not true if the measure of Ω is infinite, but in general we have the following interpolation result.

LEMMA 1.10. *If* $p \le q \le r$*, then* $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega)$ *and*

$$\|f\|_q \le \|f\|_p^\theta \|f\|_r^{1-\theta}$$

where $0 \le \theta \le 1$ *is given by*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

PROOF. Assume without loss of generality that $f \ge 0$. Using Hölder's inequality with exponents $1/\sigma$ and $1/(1-\sigma)$, we get

$$\int f^q\, dx = \int f^{\theta q} f^{(1-\theta)q}\, dx \le \left(\int f^{\theta q/\sigma}\, dx\right)^\sigma \left(\int f^{(1-\theta)q/(1-\sigma)}\, dx\right)^{1-\sigma}.$$

Choosing $\sigma/\theta = q/p$, when $(1-\sigma)/(1-\theta) = q/r$, we get

$$\int f^q\, dx \le \left(\int f^p\, dx\right)^{q\theta/p} \left(\int f^r\, dx\right)^{q(1-\theta)/r}$$

and the result follows. □

It is often useful to consider local L^p spaces consisting of functions that have finite integral on compact sets.

DEFINITION 1.11. The space $L^p_{\mathrm{loc}}(\Omega)$, where $1 \le p \le \infty$, consists of functions $f : \Omega \to \mathbb{R}$ such that $f \in L^p(\Omega')$ for every open set $\Omega' \Subset \Omega$. A sequence of functions $\{f_n\}$ converges to f in $L^p_{\mathrm{loc}}(\Omega)$ if $\{f_n\}$ converges to f in $L^p(\Omega')$ for every open set $\Omega' \Subset \Omega$.

If $p < q$, then $L^q_{\mathrm{loc}}(\Omega) \hookrightarrow L^p_{\mathrm{loc}}(\Omega)$ even if the measure of Ω is infinite. Thus, $L^1_{\mathrm{loc}}(\Omega)$ is the 'largest' space of integrable functions on Ω.

EXAMPLE 1.12. Consider $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|^a}$$

where $a \in \mathbb{R}$. Then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $a < n$. To prove this, let

$$f^\epsilon(x) = \begin{cases} f(x) & \text{if } |x| > \epsilon, \\ 0 & \text{if } |x| \le \epsilon. \end{cases}$$

Then $\{f^\epsilon\}$ is monotone increasing and converges pointwise almost everywhere to f as $\epsilon \to 0^+$. For any $R > 0$, the monotone convergence theorem implies that

$$\begin{aligned}\int_{B_R(0)} f\,dx &= \lim_{\epsilon\to 0^+}\int_{B_R(0)} f^\epsilon\,dx\\ &= \lim_{\epsilon\to 0^+}\int_\epsilon^R r^{n-a-1}\,dr\\ &= \begin{cases} \infty & \text{if } n-a\le 0,\\ (n-a)^{-1}R^{n-a} & \text{if } n-a>0,\end{cases}\end{aligned}$$

which proves the result. The function f does not belong to $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ for any value of a, since the integral of f^p diverges at infinity whenever it converges at zero.

1.5. Compactness

A subset F of a metric space X is precompact if the closure of F is compact; equivalently, F is precompact if every sequence in F has a subsequence that converges in X.

The Arzelà-Ascoli theorem gives a basic criterion for compactness in function spaces: it states that a set of continuous functions on a compact metric space is precompact if and only if it is bounded and equicontinuous. We state the result explicitly for the spaces of interest here.

THEOREM 1.13. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$. A subset $\mathcal{F}$ of $C\left(\overline{\Omega}\right)$, equipped with the maximum norm, is precompact if and only if:*

(1) *there exists a constant M such that*
$$\|f\|_\infty \le M \qquad \text{for all } f\in\mathcal{F};$$
(2) *for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, x+h \in \overline{\Omega}$ and $|h| < \delta$ then*
$$|f(x+h)-f(x)| < \epsilon \qquad \text{for all } f\in\mathcal{F}.$$

The following theorem (known as the Fréchet-Kolmogorov, Kolmogorov-Riesz, or Riesz-Tamarkin theorem) gives conditions analogous to the ones in the Arzelà-Ascoli theorem for a set to be precompact in $L^p(\mathbb{R})$, namely that the set is bounded, 'tight', and L^p-equicontinuous.

THEOREM 1.14. *Let $1 \le p < \infty$. A subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$ is precompact if and only if:*

(1) *there exists M such that*
$$\|f\|_{L^p} \le M \qquad \text{for all } f\in\mathcal{F};$$
(2) *for every $\epsilon > 0$ there exists R such that*
$$\left(\int_{|x|>R} |f(x)|^p\,dx\right)^{1/p} < \epsilon \qquad \text{for all } f\in\mathcal{F}.$$
(3) *for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$,*
$$\left(\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx\right)^{1/p} < \epsilon \qquad \text{for all } f\in\mathcal{F}.$$

For a proof, see [**18**].

1.6. Averages

For $x \in \mathbb{R}^n$ and $r > 0$, let

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$

denote the open ball centered at x with radius r, and

$$\partial B_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}$$

the corresponding sphere.

The volume of the unit ball in $\mathbb{R}^n$ is given by

$$\alpha_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$$

where Γ is the Gamma function, which satisfies

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(1) = 1, \qquad \Gamma(x+1) = x\Gamma(x).$$

Thus, for example, $\alpha_2 = \pi$ and $\alpha_3 = 4\pi/3$. An integration with respect to polar coordinates shows that the area of the $(n-1)$-dimensional unit sphere is $n\alpha_n$.

We denote the average of a function $f \in L^1_{\mathrm{loc}}(\Omega)$ over a ball $B_r(x) \Subset \Omega$, or the corresponding sphere $\partial B_r(x)$, by

$$\text{(1.2)} \quad \fint_{B_r(x)} f\,dx = \frac{1}{\alpha_n r^n} \int_{B_r(x)} f\,dx, \qquad \fint_{\partial B_r(x)} f\,dS = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B_r(x)} f\,dS.$$

If f is continuous at x, then

$$\lim_{r \to 0^+} \fint_{B_r(x)} f\,dx = f(x).$$

The following result, called the Lebesgue differentiation theorem, implies that the averages of a locally integrable function converge pointwise almost everywhere to the function as the radius r shrinks to zero.

THEOREM 1.15. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then*

$$\text{(1.3)} \qquad \lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f(x)|\,dx = 0$$

pointwise almost everywhere for $x \in \mathbb{R}^n$.

A point $x \in \mathbb{R}^n$ for which (1.3) holds is called a Lebesgue point of f. For a proof of this theorem (using the Wiener covering lemma and the Hardy-Littlewood maximal function) see Folland [**7**] or Taylor [**17**].

1.7. Convolutions

DEFINITION 1.16. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are measurable function, we define the convolution $f * g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\,dy$$

provided that the integral converges pointwise almost everywhere in x.

When defined, the convolutiom product is both commutative and associative,

$$f * g = g * f, \qquad f * (g * h) = (f * g) * h.$$

In many respects, the convolution of two functions inherits the best properties of both functions.

If $f, g \in C_c(\mathbb{R}^n)$, then their convolution also belongs to $C_c(\mathbb{R}^n)$ and

$$\operatorname{spt}(f * g) \subset \operatorname{spt} f + \operatorname{spt} g.$$

If $f \in C_c(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n)$, then $f*g \in C(\mathbb{R}^n)$ is defined, however rapidly g grows at infinity, but typically it does not have compact support. If neither f nor g have compact support, we need some conditions on their growth or decay at infinity to ensure that the convolution exists. The following result, called Young's inequality, gives conditions for the convolution of L^p functions to exist and estimates its norm.

Theorem 1.17. *Suppose that* $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

If $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

The following special cases are useful to keep in mind.

Example 1.18. If $p = q = 2$ then $r = \infty$. In this case, the result follows from the Cauchy-Schwartz inequality, since for any $x \in \mathbb{R}^n$

$$\left| \int f(x-y) g(y)\, dx \right| \le \|f\|_{L^2} \|g\|_{L^2}.$$

Moreover, a density argument shows that $f * g \in C_0(\mathbb{R}^n)$: Choose $f_k, g_k \in C_c(\mathbb{R}^n)$ such that $f_k \to f$, $g_k \to g$ in $L^2(\mathbb{R}^n)$, then $f_k * g_k \in C_c(\mathbb{R}^n)$ and $f_k * g_k \to f * g$ uniformly. A similar argument is used in the proof of the Riemann-Lebesgue lemma that $\hat{f} \in C_0(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$.

Example 1.19. If $p = q = 1$, then $r = 1$, and the result follows directly from Fubini's theorem, since

$$\int \left| \int f(x-y) g(y)\, dy \right| dx \le \int |f(x-y) g(y)|\, dx dy = \left(\int |f(x)|\, dx \right) \left(\int |g(y)|\, dy \right).$$

Thus, the space $L^1(\mathbb{R}^n)$ is an algebra under the convolution product. The Fourier transform maps the convolution product of two L^1-functions to the pointwise product of their Fourier transforms.

Example 1.20. If $q = 1$, then $p = r$. Thus convolution with an integrable function $k \in L^1(\mathbb{R}^n)$, is a bounded linear map $f \mapsto k * f$ on $L^p(\mathbb{R}^n)$.

1.8. Derivatives and multi-index notation

We define the derivative of a scalar field $u : \Omega \to \mathbb{R}$ by

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right).$$

We will also denote the ith partial derivative by $\partial_i u$, the ijth derivative by $\partial_{ij} u$, and so on. The divergence of a vector field $X = (X_1, X_2, \dots, X_n) : \Omega \to \mathbb{R}^n$ is

$$\operatorname{div} X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \dots + \frac{\partial X_n}{\partial x_n}.$$

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denote the non-negative integers. An n-dimensional multi-indexis a vector $\alpha \in \mathbb{N}_0^n$, meaning that

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \qquad \alpha_i = 0, 1, 2, \dots.$$

We write

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \qquad \alpha! = \alpha_1!\alpha_2! \dots \alpha_n!.$$

We define derivatives and powers of order α by

$$\partial^\alpha = \frac{\partial}{\partial x^{\alpha_1}} \frac{\partial}{\partial x^{\alpha_2}} \dots \frac{\partial}{\partial x^{\alpha_n}}, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are multi-indices, we define the multi-index $(\alpha + \beta)$ by

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n).$$

We denote by $\chi_n(k)$ the number of multi-indices $\alpha \in \mathbb{N}_0^n$ with order $0 \le |\alpha| \le k$, and by $\tilde{\chi}_n(k)$ the number of multi-indices with order $|\alpha| = k$. Then

$$\chi_n(k) = \frac{(n+k)!}{n!k!}, \qquad \tilde{\chi}_n(k) = \frac{(n+k-1)!}{(n-1)!k!}$$

1.8.1. Taylor's theorem for functions of several variables. The multi-index notation provides a compact way to write the multinomial theorem and the Taylor expansion of a function of several variables. The multinomial expansion of a power is

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{\alpha_1 + \dots \alpha_n = k} \binom{k}{\alpha_1 \alpha_2 \dots \alpha_n} x_i^{\alpha_i} = \sum_{|\alpha| = k} \binom{k}{\alpha} x^\alpha$$

where the multinomial coefficient of a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of order $|\alpha| = k$ is given by

$$\binom{k}{\alpha} = \binom{k}{\alpha_1 \alpha_2 \dots \alpha_n} = \frac{k!}{\alpha_1!\alpha_2! \dots \alpha_n!}.$$

THEOREM 1.21. *Suppose that $u \in C^k(B_r(x))$ and $h \in B_r(0)$. Then*

$$u(x+h) = \sum_{|\alpha| \le k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h)$$

where the remainder is given by

$$R_k(x,h) = \sum_{|\alpha| = k} \frac{\partial^\alpha u(x + \theta h)}{\alpha!} h^\alpha$$

for some $0 < \theta < 1$.

PROOF. Let $f(t) = u(x + th)$ for $0 \le t \le 1$. Taylor's theorem for a function of a single variable implies that

$$f(1) = \sum_{j=0}^{k-1} \frac{1}{j!} \frac{d^j f}{dt^j}(0) + \frac{1}{k!} \frac{d^k f}{dt^k}(\theta)$$

for some $0 < \theta < 1$. By the chain rule,

$$\frac{df}{dt} = Du \cdot h = \sum_{i=1}^{n} h_i \partial_i u,$$

and the multinomial theorem gives

$$\frac{d^k}{dt^k} = \left(\sum_{i=1}^{n} h_i \partial_i \right)^k = \sum_{|\alpha|=k} \binom{n}{\alpha} h^\alpha \partial^\alpha.$$

Using this expression to rewrite the Taylor series for f in terms of u, we get the result. □

A function $u : \Omega \to \mathbb{R}$ is real-analytic in an open set Ω if it has a power-series expansion that converges to the function in a ball of non-zero radius about every point of its domain. We denote by $C^\omega(\Omega)$ the space of real-analytic functions on Ω. A real-analytic function is C^∞, since its Taylor series can be differentiated term-by-term, but a C^∞ function need not be real-analytic. For example, see (1.4) below.

1.9. Mollifiers

The function

$$\eta(x) = \begin{cases} C \exp\left[-1/(1 - |x|^2)\right] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases} \tag{1.4}$$

belongs to $C_c^\infty(\mathbb{R}^n)$ for any constant C. We choose C so that

$$\int_{\mathbb{R}^n} \eta \, dx = 1$$

and for any $\epsilon > 0$ define the function

$$\eta^\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right). \tag{1.5}$$

Then η^ϵ is a C^∞-function with integral equal to one whose support is the closed ball $\overline{B}_\epsilon(0)$. We refer to (1.5) as the 'standard mollifier.'

We remark that $\eta(x)$ in (1.4) is not real-analytic when $|x| = 1$. All of its derivatives are zero at those points, so the Taylor series converges to zero in any neighborhood, not to the original function. The only function that is real-analytic with compact support is the zero function. In rough terms, an analytic function is a single 'organic' entity: its values in, for example, a single open ball determine its values everywhere in a maximal domain of analyticity (which is a Riemann surface) through analytic continuation. The behavior of C^∞-function at one point is, however, completely unrelated to its behavior at another point.

Suppose that $f \in L^1_{\text{loc}}(\Omega)$ is a locally integrable function. For $\epsilon > 0$, let

$$\Omega^\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\} \tag{1.6}$$

and define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ by

$$f^\epsilon(x) = \int_\Omega \eta^\epsilon(x - y) f(y)\, dy \tag{1.7}$$

where η^ϵ is the mollifier in (1.5). We define f^ϵ for $x \in \Omega^\epsilon$ so that $B_\epsilon(x) \subset \Omega$ and we have room to average f. If $\Omega = \mathbb{R}^n$, we have simply $\Omega^\epsilon = \mathbb{R}^n$. The function f^ϵ is a smooth approximation of f.

THEOREM 1.22. *Suppose that $f \in L^p_{\mathrm{loc}}(\Omega)$ for $1 \le p < \infty$, and $\epsilon > 0$. Define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ by (1.7). Then: (a) $f^\epsilon \in C^\infty(\Omega^\epsilon)$ is smooth; (b) $f^\epsilon \to f$ pointwise almost everywhere in Ω as $\epsilon \to 0^+$; (c) $f^\epsilon \to f$ in $L^p_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0^+$.*

PROOF. The smoothness of f^ϵ follows by differentiation under the integral sign

$$\partial^\alpha f^\epsilon(x) = \int_\Omega \partial^\alpha \eta^\epsilon(x - y) f(y)\, dy$$

which may be justified by use of the dominated convergence theorem. The pointwise almost everywhere convergence (at every Lebesgue point of f) follows from the Lebesgue differentiation theorem. The convergence in L^p_{loc} follows by the approximation of f by a continuous function (for which the result is easy to prove) and the use of Young's inequality, since $\|\eta^\epsilon\|_{L^1} = 1$ is bounded independently of ϵ. □

One consequence of this theorem is that the space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$. Note that this is not true when $p = \infty$, since the uniform limit of smooth test functions is continuous.

1.9.1. Cutoff functions.

THEOREM 1.23. *Suppose that $\Omega' \Subset \Omega$ are open sets in $\mathbb{R}^n$. Then there is a function $\phi \in C_c^\infty(\Omega)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on Ω'.*

PROOF. Let $d = \operatorname{dist}(\Omega', \partial\Omega)$ and define

$$\Omega'' = \{x \in \Omega : \operatorname{dist}(x, \Omega') < d/2\}.$$

Let χ be the characteristic function of Ω'', and define $\phi = \eta^{d/2} * \chi$ where η^ϵ is the standard mollifier. Then one may verify that ϕ has the required properties. □

We refer to a function with the properties in this theorem as a cutoff function.

EXAMPLE 1.24. If $0 < r < R$ and $\Omega'' = B_r(0)$, $\Omega' = B_R(0)$ are balls in $\mathbb{R}^n$, then the corresponding cut-off function ϕ satisfies

$$|D\phi| \le \frac{C}{R - r}$$

where C is a constant that is independent of r, R.

1.9.2. Partitions of unity. Partitions of unity allow us to piece together global results from local results.

THEOREM 1.25. *Suppose that K is a compact set in $\mathbb{R}^n$ which is covered by a finite collection $\{\Omega_1, \Omega_2, \dots, \Omega_N\}$ of open sets. Then there exists a collection of functions $\{\eta_1, \eta_2, \dots, \eta_N\}$ such that $0 \le \eta_i \le 1$, $\eta_i \in C_c^\infty(\Omega_i)$, and $\sum_{i=1}^N \eta_i = 1$ on K.*

We call $\{\eta_i\}$ a partition of unity subordinate to the cover $\{\Omega_i\}$. To prove this result, we use Urysohn's lemma to construct a collection of continuous functions with the desired properties, then use mollification to obtain a collection of smooth functions.

1.10. Boundaries of open sets

When we analyze solutions of a PDE in the interior of their domain of definition, we can often consider domains that are arbitrary open sets and analyze the solutions in a sufficiently small ball. In order to analyze the behavior of solutions at a boundary, however, we typically need to assume that the boundary has some sort of smoothness. In this section, we define the smoothness of the boundary of an open set. We also explain briefly how one defines analytically the normal vector-field and the surface area measure on a smooth boundary.

In general, the boundary of an open set may be complicated. For example, it can have nonzero Lebesgue measure.

Example 1.26. Let $\{q_i : i \in \mathbb{N}\}$ be an enumeration of the rational numbers $q_i \in (0,1)$. For each $i \in \mathbb{N}$, choose an open interval $(a_i, b_i) \subset (0,1)$ that contains q_i, and let

$$\Omega = \bigcup_{i\in\mathbb{N}} (a_i, b_i).$$

The Lebesgue measure of $|\Omega| > 0$ is positive, but we can make it as small as we wish; for example, choosing $b_i - a_i = \epsilon 2^{-i}$, we get $|\Omega| \le \epsilon$. One can check that $\partial\Omega = [0,1] \setminus \Omega$. Thus, if $|\Omega| < 1$, then $\partial\Omega$ has nonzero Lebesgue measure.

Moreover, an open set, or domain, need not lie on one side of its boundary (we say that Ω lies on one side of its boundary if $\overline{\Omega}^\circ = \Omega$), and corners, cusps, or other singularities in the boundary cause analytical difficulties.

Example 1.27. The unit disc in $\mathbb{R}^2$ with the nonnegative x-axis removed,

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\} \setminus \left\{(x,0) \in \mathbb{R}^2 : 0 \le x < 1\right\},$$

does not lie on one side of its boundary.

In rough terms, the boundary of an open set is smooth if it can be 'flattened out' locally by a smooth map.

Definition 1.28. Suppose that $k \in \mathbb{N}$. A map $\phi : U \to V$ between open sets U, V in $\mathbb{R}^n$ is a C^k-diffeomorphism if it one-to-one, onto, and ϕ and ϕ^{-1} have continuous derivatives of order less than or equal to k.

Note that the derivative $D\phi(x) : \mathbb{R}^n \to \mathbb{R}^n$ of a diffeomorphism $\phi : U \to V$ is an invertible linear map for every $x \in U$.

Definition 1.29. Let Ω be a bounded open set in $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that the boundary $\partial\Omega$ is C^k, or that Ω is C^k for short, if for every $x \in \overline{\Omega}$ there is an open neighborhood $U \subset \mathbb{R}^n$ of x, an open set $V \subset \mathbb{R}^n$, and a C^k-diffeomorphism $\phi : U \to V$ such that

$$\phi(U \cap \Omega) = V \cap \{y_n > 0\}, \qquad \phi(U \cap \partial\Omega) = V \cap \{y_n = 0\}$$

where $(y_1, \dots, y_n)$ are coordinates in the image space $\mathbb{R}^n$.

If ϕ is a C^∞-diffeomorphism, then we say that the boundary is C^∞, with an analogous definition of a Lipschitz or analytic boundary.

In other words, the definition says that a C^k open set in $\mathbb{R}^n$ is an n-dimensional C^k-manifold with boundary. The maps ϕ in Definition 1.29 are coordinate charts for the manifold. It follows from the definition that Ω lies on one side of its boundary and that $\partial\Omega$ is an oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ without boundary. The standard orientation is given by the outward-pointing normal (see below).

EXAMPLE 1.30. The open set

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x > 0,\, y > \sin(1/x)\right\}$$

lies on one side of its boundary, but the boundary is not C^1 since there is no coordinate chart of the required form for the boundary points $\{(x,0) : -1 \le x \le 1\}$.

1.10.1. Open sets in the plane. A simple closed curve, or Jordan curve, Γ is a set in the plane that is homeomorphic to a circle. That is, $\Gamma = \gamma(\mathbb{T})$ is the image of a one-to-one continuous map $\gamma : \mathbb{T} \to \mathbb{R}^2$ with continuous inverse $\gamma^{-1} : \Gamma \to \mathbb{T}$. (The requirement that the inverse is continuous follows from the other assumptions.) According to the Jordan curve theorem, a Jordan curve divides the plane into two disjoint connected open sets, so that $\mathbb{R}^2 \setminus \Gamma = \Omega_1 \cup \Omega_2$. One of the sets (the 'interior') is bounded and simply connected. The interior region of a Jordan curve is called a Jordan domain.

EXAMPLE 1.31. The slit disc Ω in Example 1.27 is not a Jordan domain. For example, its boundary separates into three nonempty connected components when the point $(1,0)$ is removed, but the circle remains connected when any point is removed, so $\partial\Omega$ cannot be homeomorphic to the circle.

EXAMPLE 1.32. The interior Ω of the Koch, or 'snowflake,' curve is a Jordan domain. The Hausdorff dimension of its boundary is strictly greater than one. It is interesting to note that, despite the irregular nature of its boundary, this domain has the property that every function in $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $1 \le p < \infty$ can be extended to a function in $W^{k,p}(\mathbb{R}^2)$.

If $\gamma : \mathbb{T} \to \mathbb{R}^2$ is one-to-one, C^1, and $|D\gamma| \ne 0$, then the image of γ is the C^1 boundary of the open set which it encloses. The condition that γ is one-to-one is necessary to avoid self-intersections (for example, a figure-eight curve), and the condition that $|D\gamma| \ne 0$ is necessary in order to ensure that the image is a C^1-submanifold of $\mathbb{R}^2$.

EXAMPLE 1.33. The curve $\gamma : t \mapsto \left(t^2, t^3\right)$ is not C^1 at $t = 0$ where $D\gamma(0) = 0$.

1.10.2. Parametric representation of a boundary. If Ω is an open set in $\mathbb{R}^n$ with C^k-boundary and ϕ is a chart on a neighborhood U of a boundary point, as in Definition 1.29, then we can define a local chart

$$\Phi = (\Phi_1, \Phi_2, \dots, \Phi_{n-1}) : U \cap \partial\Omega \subset \mathbb{R}^n \to W \subset \mathbb{R}^{n-1}$$

for the boundary $\partial\Omega$ by $\Phi = (\phi_1, \phi_2, \dots, \phi_{n-1})$. Thus, $\partial\Omega$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

The boundary is parametrized locally by $x_i = \Psi_i\left(y_1, y_2, \dots, y_{n-1}\right)$ where $1 \le i \le n$ and $\Psi = \Phi^{-1} : W \to U \cap \partial\Omega$. The $(n-1)$-dimensional tangent space of $\partial\Omega$ is spanned by the vectors

$$\frac{\partial\Psi}{\partial y_1}, \frac{\partial\Psi}{\partial y_2}, \dots, \frac{\partial\Psi}{\partial y_{n-1}}.$$

The outward unit normal $\nu : \partial\Omega \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$ is orthogonal to this tangent space, and it is given locally by

$$\nu = \frac{\tilde{\nu}}{|\tilde{\nu}|}, \qquad \tilde{\nu} = \frac{\partial \Psi}{\partial y_1} \wedge \frac{\partial \Psi}{\partial y_2} \wedge \cdots \wedge \frac{\partial \Psi}{\partial y_{n-1}},$$
$$\tilde{\nu}_i = \begin{vmatrix} \partial\Psi_1/\partial y_1 & \partial\Psi_1/\partial\Psi_2 & \dots & \partial\Psi_1/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_{i-1}/\partial y_1 & \partial\Psi_{i-1}/\partial y_2 & \dots & \partial\Psi_{i-1}/\partial y_{n-1} \\ \partial\Psi_{i+1}/\partial y_1 & \partial\Psi_{i+1}/\partial y_2 & \dots & \partial\Psi_{i+1}/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_n/\partial y_1 & \partial\Psi_n/\partial y_2 & \dots & \partial\Psi_n/\partial y_{n-1} \end{vmatrix}.$$

EXAMPLE 1.34. For a three-dimensional region with two-dimensional boundary, the outward unit normal is

$$\nu = \frac{(\partial\Psi/\partial y_1) \times (\partial\Psi/\partial y_2)}{|(\partial\Psi/\partial y_1) \times (\partial\Psi/\partial y_2)|}.$$

The restriction of the Euclidean metric on $\mathbb{R}^n$ to the tangent space of the boundary gives a Riemannian metric on the boundary whose volume form defines the surface measure dS. Explicitly, the pull-back of the Euclidean metric

$$\sum_{i=1}^n dx_i^2$$

to the boundary under the mapping $x = \Psi(y)$ is the metric

$$\sum_{i=1}^n \sum_{p,q=1}^{n-1} \frac{\partial\Psi_i}{\partial y_p} \frac{\partial\Psi_i}{\partial y_q} \, dy_p dy_q.$$

The volume form associated with a Riemannian metric $\sum h_{pq} \, dy_p dy_q$ is

$$\sqrt{\det h} \, dy_1 dy_2 \dots dy_{n-1}.$$

Thus the surface measure on $\partial\Omega$ is given locally by

$$dS = \sqrt{\det\left(D\Psi^t D\Psi\right)} \, dy_1 dy_2 \dots dy_{n-1}$$

where $D\Psi$ is the derivative of the parametrization,

$$D\Psi = \begin{pmatrix} \partial\Psi_1/\partial y_1 & \partial\Psi_1/\partial y_2 & \dots & \partial\Psi_1/\partial y_{n-1} \\ \partial\Psi_2/\partial y_1 & \partial\Psi_2/\partial y_2 & \dots & \partial\Psi_2/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_n/\partial y_1 & \partial\Psi_n/\partial y_2 & \dots & \partial\Psi_n/\partial y_{n-1} \end{pmatrix}.$$

These local expressions may be combined to give a global definition of the surface integral by means of a partition of unity.

EXAMPLE 1.35. In the case of a two-dimensional surface with metric

$$ds^2 = E \, dy_1^2 + 2F \, dy_1 dy_2 + G \, dy_2^2,$$

the element of surface area is

$$dS = \sqrt{EG - F^2} \, dy_1 dy_2.$$

EXAMPLE 1.36. The two-dimensional sphere

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

is a C^∞ submanifold of $\mathbb{R}^3$. A local C^∞-parametrization of

$$U = \mathbb{S}^2 \setminus \left\{(x, 0, z) \in \mathbb{R}^3 : x \geq 0\right\}$$

is given by $\Psi : W \subset \mathbb{R}^2 \to U \subset \mathbb{S}^2$ where

$$\begin{aligned} \Psi(\theta, \phi) &= (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \\ W &= \left\{(\theta, \phi) \in \mathbb{R}^3 : 0 < \theta < 2\pi,\, 0 < \phi < \pi\right\}. \end{aligned}$$

The metric on the sphere is

$$\Psi^* \left(dx^2 + dy^2 + dz^2\right) = \sin^2\phi \, d\theta^2 + d\phi^2$$

and the corresponding surface area measure is

$$dS = \sin\phi \, d\theta d\phi.$$

The integral of a continuous function $f(x, y, z)$ over the sphere that is supported in U is then given by

$$\int_{\mathbb{S}^2} f \, dS = \int_W f\left(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi\right) \sin\phi \, d\theta d\phi.$$

We may use similar rotated charts to cover the points with $x \geq 0$ and $y = 0$.

1.10.3. Representation of a boundary as a graph. An alternative, and computationally simpler, way to represent the boundary of a smooth open set is as a graph. After rotating coordinates, if necessary, we may assume that the nth component of the normal vector to the boundary is nonzero. If $k \geq 1$, the implicit function theorem implies that we may represent a C^k-boundary as a graph

$$x_n = h\left(x_1, x_2, \dots, x_{n-1}\right)$$

where $h : W \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is in $C^k(W)$ and Ω is given locally by $x_n < h(x_1, \dots, x_{n-1})$. If the boundary is only Lipschitz, then the implicit function theorem does not apply, and it is not always possible to represent a Lipschitz boundary locally as the region lying above the graph of a Lipschitz continuous function.

If $\partial\Omega$ is C^1, then the outward normal ν is given in terms of h by

$$\nu = \frac{1}{\sqrt{1 + |Dh|^2}} \left(-\frac{\partial h}{\partial x_1}, -\frac{\partial h}{\partial x_2}, \dots, -\frac{\partial h}{\partial x_{n-1}}, 1\right)$$

and the surface area measure on $\partial\Omega$ is given by

$$dS = \sqrt{1 + |Dh|^2} \, dx_1 dx_2 \dots dx_{n-1}.$$

EXAMPLE 1.37. Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and $\partial\Omega$ the unit sphere. The upper hemisphere

$$H = \{x \in \partial\Omega : x_n > 0\}$$

is the graph of $x_n = h(x')$ where $h : D \to \mathbb{R}$ is given by

$$h(x') = \sqrt{1 - |x'|^2}, \qquad D = \left\{x' \in \mathbb{R}^{n-1} : |x'| < 1\right\}$$

and we write $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. The surface measure on H is

$$dS = \frac{1}{\sqrt{1 - |x'|^2}} \, dx'$$

and the surface integral of a function $f(x)$ over H is given by

$$\int_H f \, dS = \int_D \frac{f\left(x', h(x')\right)}{\sqrt{1 - |x'|^2}} \, dx'.$$

The integral of a function over $\partial\Omega$ may be computed in terms of such integrals by use of a partition of unity subordinate to an atlas of hemispherical charts.

1.11. Change of variables

We state a theorem for a C^1 change of variables in the Lebesgue integral. A special case is the change of variables from Cartesian to polar coordinates. For proofs, see [**7, 17**].

THEOREM 1.38. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\phi : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism of Ω onto its image $\phi(\Omega)$. If $f : \phi(\Omega) \to \mathbb{R}$ is a nonnegative Lebesgue measurable function or an integrable function, then*

$$\int_{\phi(\Omega)} f(y) \, dy = \int_\Omega f \circ \phi(x) \left|\det D\phi(x)\right| \, dx.$$

We define polar coordinates in $\mathbb{R}^n \setminus \{0\}$ by $x = ry$, where $r = |x| > 0$ and $y \in \partial B_1(0)$ is a point on the unit sphere. In these coordinates, Lebesgue measure has the representation

$$dx = r^{n-1} dr dS(y)$$

where $dS(y)$ is the surface area measure on the unit sphere. We have the following result for integration in polar coordinates.

PROPOSITION 1.39. *If $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, then*

$$\begin{aligned} \int f \, dx &= \int_0^\infty \left[\int_{\partial B_1(0)} f(x + ry) \, dS(y) \right] r^{n-1} \, dr \\ &= \int_{\partial B_1(0)} \left[\int_0^\infty f(x + ry) \, r^{n-1} \, dr \right] dS(y). \end{aligned}$$

1.12. Divergence theorem

We state the divergence (or Gauss-Green) theorem.

THEOREM 1.40. *Let $X : \overline{\Omega} \to \mathbb{R}^n$ be a $C^1(\overline{\Omega})$-vector field, and $\Omega \subset \mathbb{R}^n$ a bounded open set with C^1-boundary $\partial\Omega$. Then*

$$\int_\Omega \operatorname{div} X \, dx = \int_{\partial\Omega} X \cdot \nu \, dS.$$

To prove the theorem, we prove it for functions that are compactly supported in a half-space, show that it remains valid under a C^1 change of coordinates with the divergence defined in an appropriately invariant way, and then use a partition of unity to add the results together.

In particular, if $u, v \in C^1(\overline{\Omega})$, then an application of the divergence theorem to the vector field $X = (0, 0, \ldots, uv, \ldots, 0)$, with ith component uv, gives the integration by parts formula

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i} v\, dx + \int_{\partial\Omega} uv\nu_i\, dS.$$

The statement in Theorem 1.40 is, perhaps, the natural one from the perspective of smooth differential geometry. The divergence theorem, however, remains valid under weaker assumptions than the ones in Theorem 1.40. For example, it applies to a cube, whose boundary is not C^1, as well as to other sets with piecewise smooth boundaries.

From the perspective of geometric measure theory, a general form of the divergence theorem holds for Lipschitz vector fields (vector fields whose weak derivative belongs to L^∞) and sets of finite perimeter (sets whose characteristic function has bounded variation). The surface integral is taken over a measure-theoretic boundary with respect to $(n-1)$-dimensional Hausdorff measure, and a measure-theoretic normal exists almost everywhere on the boundary with respect to this measure [**6, 19**].