CHAPTER 2

Laplace's equation

> There can be but one option as to the beauty and utility of this analysis by Laplace; but the manner in which it has hitherto been presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.[1]

Laplace's equation is

$$\Delta u = 0$$

where the Laplacian Δ is defined in Cartesian coordinates by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We may also write $\Delta = \operatorname{div} D$. The Laplacian Δ is invariant under translations (it has constant coefficients) and orthogonal transformations of $\mathbb{R}^n$. A solution of Laplace's equation is called a harmonic function.

Laplace's equation is a linear, scalar equation. It is the prototype of an elliptic partial differential equation, and many of its qualitative properties are shared by more general elliptic PDEs. The non-homogeneous version of Laplace's equation

$$-\Delta u = f$$

is called Poisson's equation. It is convenient to include a minus sign here because Δ is a negative definite operator.

The Laplace and Poisson equations, and their generalizations, arise in many different contexts.

- Potential theory e.g. in the Newtonian theory of gravity, electrostatics, heat flow, and potential flows in fluid mechanics.
- Riemannian geometry e.g. the Laplace-Beltrami operator.
- Stochastic processes e.g. the stationary Kolmogorov equation for Brownian motion.
- Complex analysis e.g. the real and imaginary parts of an analytic function of a single complex variable are harmonic.

As with any PDE, we typically want to find solutions of the Laplace or Poisson equation that satisfy additional conditions. For example, if Ω is a bounded domain in $\mathbb{R}^n$, then the classical Dirichlet problem for Poisson's equation is to find a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

[1] Kelvin and Tait, *Treatise on Natural Philosophy*, 1879

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ are given functions. The classical Neumann problem is to find a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Here, 'classical' refers to the requirement that the functions and derivatives appearing in the problem are defined pointwise as continuous functions. Dirichlet boundary conditions specify the function on the boundary, while Neumann conditions specify the normal derivative. Other boundary conditions, such as mixed (or Robin) and oblique-derivative conditions are also of interest. Also, one may impose different types of boundary conditions on different parts of the boundary (e.g. Dirichlet on one part and Neumann on another).

Here, we mostly follow Evans [**5**] (§2.2), Gilbarg and Trudinger [**10**], and Han and Lin [**12**].

2.1. Mean value theorem

Harmonic functions have the following mean-value property which states that the average value (1.2) of the function over a ball or sphere is equal to its value at the center.

THEOREM 2.1. *Suppose that $u \in C^2(\Omega)$ is harmonic in an open set Ω and $B_r(x) \Subset \Omega$. Then*

$$u(x) = \fint_{B_r(x)} u\,dx, \qquad u(x) = \fint_{\partial B_r(x)} u\,dS. \tag{2.3}$$

PROOF. If $u \in C^2(\Omega)$ and $B_r(x) \Subset \Omega$, then the divergence theorem (Theorem 1.40) implies that

$$\begin{aligned} \int_{B_r(x)} \Delta u\,dx &= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS \\ &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x+ry)\,dS(y) \\ &= r^{n-1} \frac{\partial}{\partial r}\left[\int_{\partial B_1(0)} u(x+ry)\,dS(y)\right]. \end{aligned}$$

Dividing this equation by $\alpha_n r^n$, we find that

$$\fint_{B_r(x)} \Delta u\,dx = \frac{n}{r}\frac{\partial}{\partial r}\left[\fint_{\partial B_r(x)} u\,dS\right]. \tag{2.4}$$

It follows that if u is harmonic, then its mean value over a sphere centered at x is independent of r. Since the mean value integral at $r = 0$ is equal to $u(x)$, the mean value property for spheres follows.

The mean value property for the ball follows from the mean value property for spheres by radial integration. □

The mean value property characterizes harmonic functions and has a remarkable number of consequences. For example, harmonic functions are smooth because local averages over a ball vary smoothly as the ball moves. We will prove this result by mollification, which is a basic technique in the analysis of PDEs.

Theorem 2.2. *Suppose that $u \in C(\Omega)$ has the mean-value property (2.3). Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.*

Proof. Let $\eta^\epsilon(x) = \tilde{\eta}^\epsilon(|x|)$ be the standard, radially symmetric mollifier (1.5). If $B_\epsilon(x) \Subset \Omega$, then, using Proposition 1.39 together with the facts that the average of u over each sphere centered at x is equal to $u(x)$ and the integral of η^ϵ is one, we get

$$\begin{aligned}
(\eta^\epsilon * u)(x) &= \int_{B_\epsilon(0)} \eta^\epsilon(y) u(x-y)\, dy \\
&= \int_0^\epsilon \left[\int_{\partial B_1(0)} \eta^\epsilon(rz) u(x - rz)\, dS(z)\right] r^{n-1}\, dr \\
&= n\alpha_n \int_0^\epsilon \left[⨍_{\partial B_r(x)} u\, dS\right] \tilde{\eta}^\epsilon(r) r^{n-1}\, dr \\
&= n\alpha_n u(x) \int_0^\epsilon \tilde{\eta}^\epsilon(r) r^{n-1}\, dr \\
&= u(x) \int \eta^\epsilon(y)\, dy \\
&= u(x).
\end{aligned}$$

Thus, u is smooth since $\eta^\epsilon * u$ is smooth.

If u has the mean value property, then (2.4) shows that

$$\int_{B_r(x)} \Delta u\, dx = 0$$

for every ball $B_r(x) \Subset \Omega$. Since Δu is continuous, it follows that $\Delta u = 0$ in Ω. □

Theorems 2.1–2.2 imply that any C^2-harmonic function is C^∞. The assumption that $u \in C^2(\Omega)$ is, if fact, unnecessary: Weyl showed that if a distribution $u \in \mathcal{D}'(\Omega)$ is harmonic in Ω, then $u \in C^\infty(\Omega)$.

Note that these results say nothing about the behavior of u at the boundary of Ω, which can be nasty. The reverse implication of this observation is that the Laplace equation can take rough boundary data and immediately smooth it to an analytic function in the interior.

Example 2.3. Consider the meromorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z}.$$

The real and imaginary parts of f

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = -\frac{y}{x^2 + y^2}$$

are harmonic and C^∞ in, for example, the open unit disc

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\right\}$$

but both are unbounded as $(x, y) \to (0, 0) \in \partial\Omega$.

The boundary behavior of harmonic functions can be much worse than in this example. If $\Omega \subset \mathbb{R}^n$ is any open set, then there exists a harmonic function in Ω such that

$$\liminf_{x\to\xi} u(x) = -\infty, \qquad \limsup_{x\to\xi} u(x) = \infty$$

for all $\xi \in \partial\Omega$. One can construct such a function as a sum of harmonic functions, converging uniformly on compact subsets of Ω, whose terms have singularities on a dense subset of points on $\partial\Omega$.

It is interesting to contrast this result with the the corresponding behavior of holomorphic functions of several variables. An open set $\Omega \subset \mathbb{C}^n$ is said to be a domain of holomorphy if there exists a holomorphic function $f : \Omega \to \mathbb{C}$ which cannot be extended to a holomorphic function on a strictly larger open set. Every open set in $\mathbb{C}$ is a domain of holomorphy, but when $n \geq 2$ there are open sets in $\mathbb{C}^n$ that are not domains of holomorphy, meaning that every holomorphic function on those sets can be extended to a holomorphic function on a larger open set.

2.1.1. Subharmonic and superharmonic functions. The mean value property has an extension to functions that are not necessarily harmonic but whose Laplacian does not change sign.

Definition 2.4. Suppose that Ω is an open set. A function $u \in C^2(\Omega)$ is subharmonic if $\Delta u \geq 0$ in Ω and superharmonic if $\Delta u \leq 0$ in Ω.

A function u is superharmonic if and only if $-u$ is subharmonic, and a function is harmonic if and only if it is both subharmonic and superharmonic. A suitable modification of the proof of Theorem 2.1 gives the following mean value inequality.

Theorem 2.5. *Suppose that Ω is an open set, $B_r(x) \Subset \Omega$, and $u \in C^2(\Omega)$. If u is subharmonic in Ω, then*

$$u(x) \leq \fint_{B_r(x)} u\,dx, \qquad u(x) \leq \fint_{\partial B_r(x)} u\,dS. \tag{2.5}$$

If u is superharmonic in Ω, then

$$u(x) \geq \fint_{B_r(x)} u\,dx, \qquad u(x) \geq \fint_{\partial B_r(x)} u\,dS. \tag{2.6}$$

It follows from these inequalities that the value of a subharmonic (or superharmonic) function at the center of a ball is less (or greater) than or equal to the value of a harmonic function with the same values on the boundary. Thus, the graphs of subharmonic functions lie below the graphs of harmonic functions and the graphs of superharmonic functions lie above, which explains the terminology. The direction of the inequality ($-\Delta u \leq 0$ for subharmonic functions and $-\Delta u \geq 0$ for superharmonic functions) is more natural when the inequality is stated in terms of the positive operator $-\Delta$.

Example 2.6. The function $u(x) = |x|^4$ is subharmonic in $\mathbb{R}^n$ since $\Delta u = 4(n+2)|x|^2 \geq 0$. The function is equal to the constant harmonic function $U(x) = 1$ on the sphere $|x| = 1$, and $u(x) \leq U(x)$ when $|x| \leq 1$.

2.2. Derivative estimates and analyticity

An important feature of Laplace equation is that we can estimate the derivatives of a solution in a ball in terms of the solution on a larger ball. This feature is closely connected with the smoothing properties of the Laplace equation.

THEOREM 2.7. *Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any $1 \leq i \leq n$,*

$$|\partial_i u(x)| \leq \frac{n}{r} \max_{\overline{B}_r(x)} |u|.$$

PROOF. Since u is smooth, differentiation of Laplace's equation with respect to x_i shows that $\partial_i u$ is harmonic, so by the mean value property for balls and the divergence theorem

$$\partial_i u = \fint_{B_r(x)} \partial_i u \, dx = \frac{1}{\alpha_n r^n} \int_{\partial B_r(x)} u \nu_i \, dS.$$

Taking the absolute value of this equation and using the estimate

$$\left| \int_{\partial B_r(x)} u \nu_i \, dS \right| \leq n \alpha_n r^{n-1} \max_{\overline{B}_r(x)} |u|$$

we get the result. □

One consequence of Theorem 2.7 is that a bounded harmonic function on $\mathbb{R}^n$ is constant; this is an n-dimensional extension of Liouville's theorem for bounded entire functions.

COROLLARY 2.8. *If $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in $\mathbb{R}^n$, then u is constant.*

PROOF. If $|u| \leq M$ on $\mathbb{R}^n$, then Theorem 2.7 implies that

$$|\partial_i u(x)| \leq \frac{Mn}{r}$$

for any $r > 0$. Taking the limit as $r \to \infty$, we conclude that $Du = 0$, so u is constant. □

Next we extend the estimate in Theorem 2.7 to higher-order derivatives. We use a somewhat tricky argument that gives sharp enough estimates to prove analyticity.

THEOREM 2.9. *Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any multi-index $\alpha \in \mathbb{N}_0^n$ of order $k = |\alpha|$*

$$|\partial^\alpha u(x)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x)} |u|.$$

PROOF. We prove the result by induction on $|\alpha| = k$. From Theorem 2.7, the result is true when $k = 1$. Suppose that the result is true when $|\alpha| = k$. If $|\alpha| = k + 1$, we may write $\partial^\alpha = \partial_i \partial^\beta$ where $1 \leq i \leq n$ and $|\beta| = k$. For $0 < \theta < 1$, let

$$\rho = (1 - \theta) r.$$

Then, since $\partial^\beta u$ is harmonic and $B_\rho(x) \Subset \Omega$, Theorem 2.7 implies that

$$|\partial^\alpha u(x)| \leq \frac{n}{\rho} \max_{\overline{B}_\rho(x)} \left| \partial^\beta u \right|.$$

Suppose that $y \in B_\rho(x)$. Then $B_{r-\rho}(y) \subset B_r(x)$, and using the induction hypothesis we get

$$\left|\partial^\beta u(y)\right| \le \frac{n^k e^{k-1} k!}{(r-\rho)^k} \max_{\overline{B}_{r-\rho}(y)} |u| \le \frac{n^k e^{k-1} k!}{r^k \theta^k} \max_{\overline{B}_r(x)} |u|.$$

It follows that

$$|\partial^\alpha u(x)| \le \frac{n^{k+1} e^{k-1} k!}{r^{k+1}\theta^k(1-\theta)} \max_{\overline{B}_r(x)} |u|.$$

Choosing $\theta = k/(k+1)$ and using the inequality

$$\frac{1}{\theta^k(1-\theta)} = \left(1+\frac{1}{k}\right)^k (k+1) \le e(k+1)$$

we get

$$|\partial^\alpha u(x)| \le \frac{n^{k+1} e^k (k+1)!}{r^{k+1}} \max_{\overline{B}_r(x)} |u|.$$

The result follows by induction. □

A consequence of this estimate is that the Taylor series of u converges to u near any point. Thus, we have the following result.

THEOREM 2.10. *If $u \in C^2(\Omega)$ is harmonic in an open set Ω then u is real-analytic in Ω.*

PROOF. Suppose that $x \in \Omega$ and choose $r > 0$ such that $B_{2r}(x) \Subset \Omega$. Since $u \in C^\infty(\Omega)$, we may expand it in a Taylor series with remainder of any order $k \in \mathbb{N}$ to get

$$u(x+h) = \sum_{|\alpha| \le k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h),$$

where we assume that $|h| < r$. From Theorem 1.21, the remainder is given by

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha \tag{2.7}$$

for some $0 < \theta < 1$.

To estimate the remainder, we use Theorem 2.9 to get

$$|\partial^\alpha u(x+\theta h)| \le \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x+\theta h)} |u|.$$

Since $|h| < r$, we have $B_r(x+\theta h) \subset B_{2r}(x)$, so for any $0 < \theta < 1$ we have

$$\max_{\overline{B}_r(x+\theta h)} |u| \le M, \qquad M = \max_{\overline{B}_{2r}(x)} |u|.$$

It follows that

$$|\partial^\alpha u(x+\theta h)| \le \frac{M n^k e^{k-1} k!}{r^k}. \tag{2.8}$$

Since $|h^\alpha| \le |h|^k$ when $|\alpha| = k$, we get from (2.7) and (2.8) that

$$|R_k(x,h)| \le \frac{M n^k e^{k-1} |h|^k k!}{r^k} \left(\sum_{|\alpha|=k} \frac{1}{\alpha!}\right).$$

The multinomial expansion

$$n^k = (1 + 1 + \cdots + 1)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} = \sum_{|\alpha|=k} \frac{k!}{\alpha!}$$

shows that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{n^k}{k!}.$$

Therefore, we have

$$|R_k(x,h)| \leq \frac{M}{e} \left(\frac{n^2 e|h|}{r}\right)^k.$$

Thus $R_k(x,h) \to 0$ as $k \to \infty$ if

$$|h| < \frac{r}{n^2 e},$$

meaning that the Taylor series of u at any $x \in \Omega$ converges to u in a ball of non-zero radius centered at x. □

It follows that, as for analytic functions, the global values of a harmonic function is determined its values in arbitrarily small balls (or by the germ of the function at a single point).

Corollary 2.11. *Suppose that u, v are harmonic in a connected open set $\Omega \subset \mathbb{R}^n$ and $\partial^\alpha u(\bar{x}) = \partial^\alpha v(\bar{x})$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ at some point $\bar{x} \in \Omega$. Then $u = v$ in Ω.*

Proof. Let

$$F = \{x \in \Omega : \partial^\alpha u(x) = \partial^\alpha v(x) \text{ for all } \alpha \in \mathbb{N}_0^n\}.$$

Then $F \neq \emptyset$, since $\bar{x} \in F$, and F is closed in Ω, since

$$F = \bigcap_{\alpha \in \mathbb{N}_0^n} \left[\partial^\alpha (u - v)\right]^{-1}(0)$$

is an intersection of relatively closed sets. Theorem 2.10 implies that if $x \in F$, then the Taylor series of u, v converge to the same value in some ball centered at x. Thus u, v and all of their partial derivatives are equal in this ball, so F is open. Since Ω is connected, it follows that $F = \Omega$. □

A physical explanation of this property is that Laplace's equation describes an equilibrium solution obtained from a time-dependent solution in the limit of infinite time. For example, in heat flow, the equilibrium is attained as the result of thermal diffusion across the entire domain, while an electrostatic field is attained only after all non-equilibrium electric fields propagate away as electromagnetic radiation. In this infinite-time limit, a change in the field near any point influences the field everywhere else, and consequently complete knowledge of the solution in an arbitrarily small region carries information about the solution in the entire domain.

Although, in principle, a harmonic function function is globally determined by its local behavior near any point, the reconstruction of the global behavior is sensitive to small errors in the local behavior.

EXAMPLE 2.12. Let $\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\, y \in \mathbb{R}\right\}$ and consider for $n \in \mathbb{N}$ the function

$$u_n(x,y) = ne^{-nx} \sin ny,$$

which is harmonic. Then

$$\partial_y^k u_n(x,1) = (-1)^k n^{k+1} e^{-n} \sin nx$$

converges uniformly to zero as $n \to \infty$ for any $k \in \mathbb{N}_0$. Thus, u_n and any finite number of its derivatives are arbitrarily close to zero at $x = 1$ when n is sufficiently large. Nevertheless, $u_n(0,y) = n \sin(ny)$ is arbitrarily large at $y = 0$.

2.3. Maximum principle

The maximum principle states that a non-constant harmonic function cannot attain a maximum (or minimum) at an interior point of its domain. This result implies that the values of a harmonic function in a bounded domain are bounded by its maximum and minimum values on the boundary. Such maximum principle estimates have many uses, but they are typically available only for scalar equations, not systems of PDEs.

THEOREM 2.13. *Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is subharmonic and attains a global maximum value in Ω, then u is constant in Ω.*

PROOF. By assumption, u is bounded from above and attains its maximum in Ω. Let

$$M = \max_\Omega u,$$

and consider

$$F = u^{-1}\left(\{M\}\right) = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty and relatively closed in Ω since u is continuous. (A subset F is relatively closed in Ω if $F = \tilde{F} \cap \Omega$ where $\tilde{F}$ is closed in $\mathbb{R}^n$.) If $x \in F$ and $B_r(x) \Subset \Omega$, then the mean value inequality (2.5) for subharmonic functions implies that

$$\fint_{B_r(x)} [u(y) - u(x)]\, dy = \fint_{B_r(x)} u(y)\, dy - u(x) \geq 0.$$

Since u attains its maximum at x, we have $u(y) - u(x) \leq 0$ for all $y \in \Omega$, and it follows that $u(y) = u(x)$ in $B_r(x)$. Therefore F is open as well as closed. Since Ω is connected, and F is nonempty, we must have $F = \Omega$, so u is constant in Ω. □

If Ω is not connected, then u is constant in any connected component of Ω that contains an interior point where u attains a maximum value.

EXAMPLE 2.14. The function $u(x) = |x|^2$ is subharmonic in $\mathbb{R}^n$. It attains a global minimum in $\mathbb{R}^n$ at the origin, but it does not attain a global maximum in any open set $\Omega \subset \mathbb{R}^n$. It does, of course, attain a maximum on any bounded closed set $\overline{\Omega}$, but the attainment of a maximum at a boundary point instead of an interior point does not imply that a subharmonic function is constant.

It follows immediately that superharmonic functions satisfy a minimum principle, and harmonic functions satisfy a maximum and minimum principle.

THEOREM 2.15. *Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is harmonic and attains either a global minimum or maximum in Ω, then u is constant.*

PROOF. Any superharmonic function u that attains a minimum in Ω is constant, since $-u$ is subharmonic and attains a maximum. A harmonic function is both subharmonic and superharmonic. $\square$

EXAMPLE 2.16. The function

$$u(x,y) = x^2 - y^2$$

is harmonic in $\mathbb{R}^2$ (it's the real part of the analytic function $f(z) = z^2$). It has a critical point at 0, meaning that $Du(0) = 0$. This critical point is a saddle-point, however, not an extreme value. Note also that

$$\fint_{B_r(0)} u\, dxdy = \frac{1}{2\pi}\int_0^{2\pi} \left(\cos^2\theta - \sin^2\theta\right)\, d\theta = 0$$

as required by the mean value property.

One consequence of this property is that any nonconstant harmonic function is an open mapping, meaning that it maps opens sets to open sets. This is not true of smooth functions such as $x \mapsto |x|^2$ that attain an interior extreme value.

2.3.1. The weak maximum principle. Theorem 2.13 is an example of a strong maximum principle, because it states that a function which attains an interior maximum is a trivial constant function. This result leads to a weak maximum principle for harmonic functions, which states that the function is bounded inside a domain by its values on the boundary. A weak maximum principle does not exclude the possibility that a non-constant function attains an interior maximum (although it implies that an interior maximum value cannot exceed the maximum value of the function on the boundary).

THEOREM 2.17. *Suppose that Ω is a bounded, connected open set in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \qquad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

PROOF. Since u is continuous and $\overline{\Omega}$ is compact, u attains its global maximum and minimum on $\overline{\Omega}$. If u attains a maximum or minimum value at an interior point, then u is constant by Theorem 2.15, otherwise both extreme values are attained on the boundary. In either case, the result follows. $\square$

Let us give a second proof of this theorem that does not depend on the mean value property. Instead, we use an argument based on the non-positivity of the second derivative at an interior maximum. In the proof, we need to account for the possibility of degenerate maxima where the second derivative is zero.

PROOF. For $\epsilon > 0$, let

$$u^\epsilon(x) = u(x) + \epsilon|x|^2.$$

Then $\Delta u^\epsilon = 2n\epsilon > 0$ since u is harmonic. If u^ϵ attained a local maximum at an interior point, then $\Delta u^\epsilon \le 0$ by the second derivative test. Thus u^ϵ has no interior maximum, and it attains its maximum on the boundary. If $|x| \le R$ for all $x \in \Omega$, it follows that

$$\sup_\Omega u \le \sup_\Omega u^\epsilon \le \sup_{\partial\Omega} u^\epsilon \le \sup_{\partial\Omega} u + \epsilon R^2.$$

Letting $\epsilon \to 0^+$, we get that $\sup_\Omega u \leq \sup_{\partial\Omega} u$. An application of the same argument to $-u$ gives $\inf_\Omega u \geq \inf_{\partial\Omega} u$, and the result follows. □

Subharmonic functions satisfy a maximum principle, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, while superharmonic functions satisfy a minimum principle $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

The conclusion of Theorem 2.17 may also be stated as

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u \qquad \text{for all } x \in \Omega.$$

In physical terms, this means for example that the interior of a bounded region which contains no heat sources or sinks cannot be hotter than the maximum temperature on the boundary or colder than the minimum temperature on the boundary.

The maximum principle gives a uniqueness result for the Dirichlet problem for the Poisson equation.

THEOREM 2.18. *Suppose that Ω is a bounded, connected open set in $\mathbb{R}^n$ and $f \in C(\Omega)$, $g \in C(\partial\Omega)$ are given functions. Then there is at most one solution of the Dirichlet problem (2.1) with $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

PROOF. Suppose that $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy (2.1). Let $v = u_1 - u_2$. Then $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω and $v = 0$ on $\partial\Omega$. The maximum principle implies that $v = 0$ in Ω, so $u_1 = u_2$, and a solution is unique. □

This theorem, of course, does not address the question of whether such a solution exists. In general, the stronger the conditions we impose upon a solution, the easier it is to show uniqueness and the harder it is to prove existence. When we come to prove an existence theorem, we will begin by showing the existence of weaker solutions e.g. solutions in $H^1(\Omega)$ instead of $C^2(\Omega)$. We will then show that these solutions are smooth under suitable assumptions on f, g, and Ω.

2.3.2. Hopf's proof of the maximum principle. Next, we give an alternative proof of the strong maximum principle Theorem 2.13 due to E. Hopf.[2] This proof does not use the mean value property and it works for other elliptic PDEs, not just the Laplace equation.

PROOF. As before, let $M = \max_{\overline{\Omega}} u$ and define

$$F = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty by assumption, and it is relatively closed in Ω since u is continuous.

Now suppose, for contradiction, that $F \neq \Omega$. Then

$$G = \Omega \setminus F$$

is nonempty and open, and the boundary $\partial F \cap \Omega = \partial G \cap \Omega$ is nonempty (otherwise F, G are open and Ω is not connected).

Choose $y \in \partial G \cap \Omega$ and let $d = \operatorname{dist}(y, \partial\Omega) > 0$. There exist points in G that are arbitrarily close to y, so we may choose $x \in G$ such that $|x - y| < d/2$. If

[2] There were two Hopf's (at least): Eberhard Hopf (1902–1983) is associated with the Hopf maximum principle (1927), the Hopf bifurcation theorem, the Wiener-Hopf method in integral equations, and the Cole-Hopf transformation for solving Burgers equation; Heinz Hopf (1894–1971) is associated with the Hopf-Rinow theorem in Riemannian geometry, the Hopf fibration in topology, and Hopf algebras.

$r = \operatorname{dist}(x, F)$, it follows that $0 < r < d/2$, so $\overline{B}_r(x) \subset G$. Moreover, there exists at least one point $\bar{x} \in \partial B_r(x) \cap \partial G$ such that $u(\bar{x}) = M$.

We therefore have the following situation: u is subharmonic in an open set G where $u < M$, the ball $B_r(x)$ is contained in G, and $u(\bar{x}) = M$ for some point $\bar{x} \in \partial B_r(x) \cap \partial G$. The Hopf boundary point lemma, proved below, then implies that

$$\partial_\nu u(\bar{x}) > 0,$$

where ∂_ν is the outward unit normal derivative to the sphere $\partial B_r(z)$

However, since $\bar{x}$ is an interior point of Ω and u attains its maximum value M there, we have $Du(\bar{x}) = 0$, so

$$\partial_\nu u(\bar{x}) = Du(\bar{x}) \cdot \nu = 0.$$

This contradiction proves the theorem. □

Before proving the Hopf lemma, we make a definition.

Definition 2.19. An open set Ω satisfies the interior sphere condition at $\bar{x} \in \partial\Omega$ if there is an open ball $B_r(x)$ contained in Ω such that $\bar{x} \in \partial B_r(x)$

The interior sphere condition is satisfied by open sets with a C^2-boundary, but — as the following example illustrates — it need not be satisfied by open sets with a C^1-boundary, and in that case the conclusion of the Hopf lemma may not hold.

Example 2.20. Let

$$u = \Re\left(\frac{z}{\log z}\right) = \frac{x \log r - y\theta}{\log^2 r + \theta^2}$$

where $\log z = \log r + i\theta$ with $-\pi/2 < \theta < \pi/2$. Define

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\, u(x, y) < 0\right\}.$$

Then u is harmonic in Ω, since f is analytic there, and $\partial\Omega$ is C^1 near the origin, with unit outward normal $(-1, 0)$ at the origin. The curvature of $\partial\Omega$, however, becomes infinite at the origin, and the interior sphere condition fails. Moreover, the normal derivative $\partial_\nu u(0, 0) = -u_x(0, 0) = 0$ vanishes at the origin, and it is not strictly positive as would be required by the Hopf lemma.

Lemma 2.21. *Suppose that $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ is subharmonic in an open set Ω and $u(x) < M$ for every $x \in \Omega$. If $u(\bar{x}) = M$ for some $\bar{x} \in \partial\Omega$ and Ω satisfies the interior sphere condition at $\bar{x}$, then $\partial_\nu u(\bar{x}) > 0$, where ∂_ν is the derivative in the outward unit normal direction to a sphere that touches $\partial\Omega$ at $\bar{x}$.*

Proof. We want to perturb u to $u^\epsilon = u + \epsilon v$ by a function ϵv with strictly negative normal derivative at $\bar{x}$, while preserving the conditions that $u^\epsilon(\bar{x}) = M$, u^ϵ is subharmonic, and $u^\epsilon < M$ near $\bar{x}$. This will imply that the normal derivative of u at $\bar{x}$ is strictly positive.

We first construct a suitable perturbing function v. Given a ball $B_R(x)$, we want $v \in C^2(\mathbb{R}^n)$ to have the following properties:

(1) $v = 0$ on $\partial B_R(x)$;
(2) $v = 1$ on $\partial B_{R/2}(x)$;
(3) $\partial_\nu v < 0$ on $\partial B_R(x)$;
(4) $\Delta v \geq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$.

We consider without loss of generality a ball $B_R(0)$ centered at 0. Thus, we want to construct a subharmonic function in the annular region $R/2 < |x| < R$ which is 1 on the inner boundary and 0 on the outer boundary, with strictly negative outward normal derivative.

The harmonic function that is equal to 1 on $|x| = R/2$ and 0 on $|x| = R$ is given by

$$u(x) = \frac{1}{2^{n-2}-1}\left[\left(\frac{R}{|x|}\right)^{n-2} - 1\right]$$

(We assume that $n \geq 3$ for simplicity.) Note that

$$\partial_\nu u = -\frac{n-2}{2^{n-2}-1}\frac{1}{R} < 0 \qquad \text{on } |x| = R,$$

so we have room to fit a subharmonic function beneath this harmonic function while preserving the negative normal derivative.

Explicitly, we look for a subharmonic function of the form

$$v(x) = c\left[e^{-\alpha|x|^2} - e^{-\alpha R^2}\right]$$

where c, α are suitable positive constants. We have $v(x) = 0$ on $|x| = R$, and choosing

$$c = \frac{1}{e^{-\alpha R^2/4} - e^{-\alpha R^2}},$$

we have $v(R/2) = 1$. Also, $c > 0$ for $\alpha > 0$. The outward normal derivative of v is the radial derivative, so

$$\partial_\nu v(x) = -2c\alpha|x|e^{-\alpha|x|^2} < 0 \qquad \text{on } |x| = R.$$

Finally, using the expression for the Laplacian in polar coordinates, we find that

$$\Delta v(x) = 2c\alpha\left[2\alpha|x|^2 - n\right]e^{-\alpha|x|^2}.$$

Thus, choosing $\alpha \geq 2n/R^2$, we get $\Delta v < 0$ for $R/2 < |x| < R$, and this gives a function v with the required properties.

By the interior sphere condition, there is a ball $B_R(x) \subset \Omega$ with $\bar{x} \in \partial B_R(x)$. Let

$$M' = \max_{\overline{B}_{R/2}(x)} u < M$$

and define $\epsilon = M - M' > 0$. Let

$$w = u + \epsilon v - M.$$

Then $w \leq 0$ on $\partial B_R(x)$ and $\partial B_{R/2}(x)$ and $\Delta w \geq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$. The maximum principle for subharmonic functions implies that $w \leq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$. Since $w(\bar{x}) = 0$, it follows that $\partial_\nu w(\bar{x}) \geq 0$. Therefore

$$\partial_\nu u(\bar{x}) = \partial_\nu w(\bar{x}) - \epsilon\partial_\nu v(\bar{x}) > 0,$$

which proves the result. $\square$

2.4. Harnack's inequality

The maximum principle gives a basic pointwise estimate for solutions of Laplace's equation, and it has a natural physical interpretation. Harnack's inequality is another useful pointwise estimate, although its physical interpretation is less obvious. It states that if a function is nonnegative and harmonic in a domain, then the ratio of the maximum and minimum of the function on a compactly supported subdomain is bounded by a constant that depends only on the domains. This inequality controls, for example, the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function.

THEOREM 2.22. *Suppose that $\Omega' \Subset \Omega$ is a connected open set that is compactly contained an open set Ω. There exists a constant C, depending only on Ω and Ω', such that if $u \in C(\Omega)$ is a non-negative function with the mean value property, then*

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u. \tag{2.9}$$

PROOF. First, we establish the inequality for a compactly contained open ball. Suppose that $x \in \Omega$ and $B_{4R}(x) \subset \Omega$, and let u be any non-negative function with the mean value property in Ω. If $y \in B_R(x)$, then,

$$u(y) = \fint_{B_R(y)} u\, dx \le 2^n \fint_{B_{2R}(x)} u\, dx$$

since $B_R(y) \subset B_{2R}(x)$ and u is non-negative. Similarly, if $z \in B_R(x)$, then

$$u(z) = \fint_{B_{3R}(z)} u\, dx \ge \left(\frac{2}{3}\right)^n \fint_{B_{2R}(x)} u\, dx$$

since $B_{3R}(z) \supset B_{2R}(x)$. It follows that

$$\sup_{B_R(x)} u \le 3^n \inf_{B_R(x)} u.$$

Suppose that $\Omega' \Subset \Omega$ and $0 < 4R < \operatorname{dist}(\Omega', \partial\Omega)$. Since $\overline{\Omega'}$ is compact, we may cover Ω' by a finite number of open balls of radius R, where the number N of such balls depends only on Ω' and Ω. Moreover, since Ω' is connected, for any $x, y \in \Omega$ there is a sequence of at most N overlapping balls $\{B_1, B_2, \dots, B_k\}$ such that $B_i \cap B_{i+1} \ne \emptyset$ and $x \in B_1$, $y \in B_k$. Applying the above estimate to each ball and combining the results, we obtain that

$$\sup_{\Omega'} u \le 3^{nN} \inf_{\Omega'} u.$$

□

In particular, it follows from (2.9) that for any $x, y \in \Omega'$, we have

$$\frac{1}{C} u(y) \le u(x) \le C u(y).$$

Harnack's inequality has strong consequences. For example, it implies that if $\{u_n\}$ is a decreasing sequence of harmonic functions in Ω and $\{u_n(x)\}$ is bounded for some $x \in \Omega$, then the sequence converges uniformly on compact subsets of Ω to a function that is harmonic in Ω. By contrast, the convergence of an arbitrary sequence of smooth functions at a single point in no way implies its convergence anywhere else, nor does uniform convergence of smooth functions imply that their limit is smooth.

You can compare this situation with what happens for analytic functions in complex analysis. If $\{f_n\}$ is a sequence of analytic functions

$$f_n : \Omega \subset \mathbb{C} \to \mathbb{C}$$

that converges uniformly on compact subsets of Ω to a function f, then f is also analytic in Ω because uniform convergence implies that the Cauchy integral formula continues to hold for f, and differentiation of this formula implies that f is analytic.

2.5. Green's identities

Green's identities provide the main energy estimates for the Laplace and Poisson equations.

THEOREM 2.23. *If Ω is a bounded C^1 open set in $\mathbb{R}^n$ and $u, v \in C^2(\overline{\Omega})$, then*

$$\int_\Omega u\Delta v\, dx = -\int_\Omega Du \cdot Dv\, dx + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\, dS, \tag{2.10}$$

$$\int_\Omega u\Delta v\, dx = \int_\Omega v\Delta u\, dx + \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)\, dS. \tag{2.11}$$

PROOF. Integrating the identity

$$\operatorname{div}(uDv) = u\Delta v + Du \cdot Dv$$

over Ω and using the divergence theorem, we get (2.10). Integrating the identity

$$\operatorname{div}(uDv - vDu) = u\Delta v - v\Delta u,$$

we get (2.11). □

Equations (2.10) and (2.11) are Green's first and second identity, respectively. The second Green's identity implies that the Laplacian Δ is a formally self-adjoint differential operator.

Green's first identity provides a proof of the uniqueness of solutions of the Dirichlet problem based on estimates of L^2-norms of derivatives instead of maximum norms. Such integral estimates are called energy estimates, because in many (though not all) cases these integral norms may be interpreted physically as the energy of a solution.

THEOREM 2.24. *Suppose that Ω is a connected, bounded C^1 open set, $f \in C(\overline{\Omega})$, and $g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solution of the Dirichlet problem (2.1), then $u_1 = u_2$; and if $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions of the Neumann problem (2.2), then $u_1 = u_2 + C$ where $C \in \mathbb{R}$ is a constant.*

PROOF. Let $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω and either $w = 0$ or $\partial w/\partial\nu = 0$ on $\partial\Omega$. Setting $u = w$, $v = w$ in (2.10), it follows that the boundary integral and the integral $\int_\Omega w\Delta w\, dx$ vanish, so that

$$\int_\Omega |Dw|^2\, dx = 0.$$

Therefore $Dw = 0$ in Ω, so w is constant. For the Dirichlet problem, $w = 0$ on $\partial\Omega$ so the constant is zero, and both parts of the result follow. □

2.6. Fundamental solution

We define the fundamental solution or free-space Green's function $\Gamma : \mathbb{R}^n \to \mathbb{R}$ (not to be confused with the Gamma function!) of Laplace's equation by

$$\begin{aligned} \Gamma(x) &= \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}} \qquad \text{if } n \geq 3, \\ \Gamma(x) &= -\frac{1}{2\pi} \log |x| \qquad\qquad \text{if } n = 2. \end{aligned} \tag{2.12}$$

The corresponding potential for $n = 1$ is

$$\Gamma(x) = -\frac{1}{2}|x|, \tag{2.13}$$

but we will consider only the multi-variable case $n \geq 2$. (Our sign convention for Γ is the same as Evans [**5**], but the opposite of Gilbarg and Trudinger [**10**].)

2.6.1. Properties of the solution. The potential $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is smooth away from the origin. For $x \neq 0$, we compute that

$$\partial_i \Gamma(x) = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}} \frac{x_i}{|x|}, \tag{2.14}$$

and

$$\partial_{ii}\Gamma(x) = \frac{1}{\alpha_n} \frac{x_i^2}{|x|^{n+2}} - \frac{1}{n\alpha_n} \frac{1}{|x|^n}.$$

It follows that

$$\Delta\Gamma = 0 \qquad \text{if } x \neq 0,$$

so Γ is harmonic in any open set that does not contain the origin. The function Γ is homogeneous of degree $-n+2$, its first derivative is homogeneous of degree $-n+1$, and its second derivative is homogeneous of degree n.

From (2.14), we have for $x \neq 0$ that

$$D\Gamma \cdot \frac{x}{|x|} = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}}$$

Thus we get the following surface integral over a sphere centered at the origin with normal $\nu = x/|x|$:

$$-\int_{\partial B_r(0)} D\Gamma \cdot \nu \, dS = 1. \tag{2.15}$$

As follows from the divergence theorem and the fact that Γ is harmonic in $B_R(0) \setminus B_r(0)$, this integral does not depend on r. The surface integral is not zero, however, as it would be for a function that was harmonic everywhere inside $B_r(0)$, including at the origin. The normalization of the flux integral in (2.15) to one accounts for the choice of the multiplicative constant in the definition of Γ.

The function Γ is unbounded as $x \to 0$ with $\Gamma(x) \to \infty$. Nevertheless, Γ and $D\Gamma$ are locally integrable. For example, the local integrability of $\partial_i \Gamma$ in (2.14) follows from the estimate

$$|\partial_i \Gamma(x)| \leq \frac{C_n}{|x|^{n-1}},$$

since $|x|^{-a}$ is locally integrable on $\mathbb{R}^n$ when $a < n$ (see Example 1.12). The second partial derivatives of Γ are not locally integrable, however, since they are of the order $|x|^{-n}$ as $x \to 0$.

2.6.2. Physical interpretation. Suppose, as in electrostatics, that u is the potential due to a charge distribution with smooth density f and $E = -Du$ is the electric field. Since $-\Delta u = f$, the divergence theorem implies that the flux of E through a boundary $\partial\Omega$ is equal to the to charge inside the enclosed volume, since

$$\int_{\partial\Omega} E \cdot \nu \, dS = \int_{\Omega} (-\Delta u) \; dx = \int_{\Omega} f \, dx.$$

Thus, since $\Delta\Gamma = 0$ for $x \neq 0$ and from (2.15) the flux of $-D\Gamma$ through any sphere centered at the origin is equal to one, we may interpret Γ as the potential due to a point charge located at the origin. In the sense of distribution, Γ satisfies the PDE

$$-\Delta\Gamma = \delta$$

where δ is the delta-function supported at the origin. We refer to such a solution as a Green's function of the Laplacian.

In three space dimensions the electric field $E = -D\Gamma$ is given by

$$E = -\frac{1}{4\pi}\frac{1}{|x|^2}\frac{x}{|x|},$$

corresponding to an inverse-square force directed away from the origin. For gravity, which is always attractive, the force has the opposite sign. This explains the connection between the Laplace and Poisson equations and Newton's inverse square law of gravitation.

As $|x| \to \infty$, the potential $\Gamma(x)$ approaches zero if $n \geq 3$, but $\Gamma(x) \to -\infty$ as $|x| \to \infty$ if $n = 2$. Physically, this corresponds to the fact that only a finite amount of energy is required to remove an object from a point source in three or more space dimensions (for example, to remove a rocket from the earth's gravitational field) but an infinite amount of energy is required to remove an object from a line source in two space dimensions.

We will use the point-source potential Γ to construct solutions of Poisson's equation for rather general right hand sides. The physical interpretation of the method is that we can obtain the potential of a general source by representing the source as a continuous distribution of point sources and superposing the corresponding point-source potential as in (2.24) below. This method, of course, depends crucially on the linearity of the equation.

2.7. The Newtonian potential

Consider the equation

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given function, which for simplicity we assume is smooth and compactly supported.

THEOREM 2.25. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and let*

$$u = \Gamma * f$$

where Γ is the fundamental solution (2.12). Then $u \in C^\infty(\mathbb{R}^n)$ and

$$-\Delta u = f. \tag{2.16}$$

PROOF. Since $f \in C_c^\infty(\mathbb{R}^n)$ and $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^n)$, Theorem 1.22 implies that $u \in C^\infty(\mathbb{R}^n)$ and

$$\Delta u = \Gamma * (\Delta f) \tag{2.17}$$

Our objective is to transfer the Laplacian across the convolution from f to Γ.

If $x \notin \operatorname{spt} f$, then we may choose a smooth open set Ω that contains $\operatorname{spt} f$ such that $x \notin \Omega$. Then $\Gamma(x-y)$ is a smooth, harmonic function of y in $\overline{\Omega}$ and f, Df are zero on $\partial\Omega$. Green's theorem therefore implies that

$$\Delta u(x) = \int_\Omega \Gamma(x-y)\Delta f(y)\,dy = \int_\Omega \Delta\Gamma(x-y) f(y)\,dy = 0,$$

which shows that $-\Delta u(x) = f(x)$.

If $x \in \operatorname{spt} f$, we must be careful about the non-integrable singularity in $\Delta\Gamma$. We therefore 'cut out' a ball of radius r about the singularity, apply Green's theorem to the resulting smooth integral, and then take the limit as $r \to 0^+$.

Let Ω be an open set that contains the support of f and define

$$\Omega_r(x) = \Omega \setminus B_r(x)\,. \tag{2.18}$$

Since Δf is bounded with compact support and Γ is locally integrable, the Lebesgue dominated convergence theorem implies that

$$\Gamma * (\Delta f)(x) = \lim_{r\to 0^+} \int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\,dy. \tag{2.19}$$

The potential $\Gamma(x-y)$ is a smooth, harmonic function of y in $\overline{\Omega_r(x)}$. Thus Green's identity (2.11) gives

$$\begin{aligned}
&\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\,dy \\
&\qquad = \int_{\partial\Omega} \left[\Gamma(x-y) D_y f(y)\cdot\nu(y) - D_y\Gamma(x-y)\cdot\nu(y) f(y)\right] dS(y) \\
&\qquad - \int_{\partial B_r(x)} \left[\Gamma(x-y) D_y f(y)\cdot\nu(y) - D_y\Gamma(x-y)\cdot\nu(y) f(y)\right] dS(y)
\end{aligned}$$

where we use the radially outward unit normal on the boundary. The boundary terms on $\partial\Omega$ vanish because f and Df are zero there, so

$$\begin{aligned}
\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\,dy = &-\int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\,dS(y) \\
&+ \int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y) f(y)\,dS(y).
\end{aligned} \tag{2.20}$$

Since Df is bounded and $\Gamma(x) = O(|x|^{n-2})$ if $n \geq 3$, we have

$$\int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\,dS(y) = O(r) \qquad \text{as } r \to 0^+.$$

The integral is $O(r\log r)$ if $n = 2$. In either case,

$$\lim_{r\to 0^+} \int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\,dS(y) = 0. \tag{2.21}$$

For the surface integral in (2.20) that involves $D\Gamma$, we write

$$\begin{aligned}\int_{\partial B_r(x)} & D_y\Gamma(x-y)\cdot\nu(y)f(y)\,dS(y) \\ &= \int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\left[f(y)-f(x)\right]\,dS(y) \\ &\quad + f(x)\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,dS(y).\end{aligned}$$

From (2.15),

$$\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,dS(y) = -1;$$

and, since f is smooth,

$$\int_{\partial B_r(x)} D_y\Gamma(x-y)\left[f(y)-f(x)\right]\,dS(y) = O\left(r^{n-1}\cdot\frac{1}{r^{n-1}}\cdot r\right)\to 0$$

as $r\to 0^+$. It follows that

$$\lim_{r\to 0^+}\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)f(y)\,dS(y) = -f(x). \tag{2.22}$$

Taking the limit of (2.20) as $r\to 0^+$ and using (2.21) and (2.22) in the result, we get

$$\lim_{r\to 0^+}\int_{\Omega_r(x)}\Gamma(x-y)\Delta f(y)\,dy = -f(x).$$

The use of this equation in (2.19) shows that

$$\Gamma * (\Delta f) = -f, \tag{2.23}$$

and the use of (2.23) in (2.17) gives (2.16). □

Equation (2.23) is worth noting: it provides a representation of a function $f\in C_c^\infty(\mathbb{R}^n)$ as a convolution of its Laplacian with the Newtonian potential.

The potential u associated with a source distribution f is given by

$$u(x) = \int \Gamma(x-y)f(y)\,dy. \tag{2.24}$$

We call u the Newtonian potential of f. We may interpret $u(x)$ as a continuous superposition of potentials proportional to $\Gamma(x-y)$ due to point sources of strength $f(y)\,dy$ located at y.

If $n\geq 3$, the potential $\Gamma * f(x)$ of a compactly supported, integrable function approaches zero as $|x|\to\infty$. We have

$$\Gamma * f(x) = \frac{1}{n(n-2)\alpha_n|x|^{n-2}}\int\left(\frac{|x|}{|x-y|}\right)^{n-2} f(y)\,dy,$$

and by the Lebesgue dominated convergence theorem,

$$\lim_{|x|\to\infty}\int\left(\frac{|x|}{|x-y|}\right)^{n-2} f(y)\,dy = \int f(y)\,dy.$$

Thus, the asymptotic behavior of the potential is the same as that of a point source whose charge is equal to the total charge of the source density f. If $n=2$, the potential, in general, grows logarithmically as $|x|\to\infty$.

If $n \geq 3$, Liouville's theorem (Corollary 2.8) implies that the Newtonian potential $\Gamma * f$ is the unique solution of $-\Delta u = f$ such that $u(x) \to 0$ as $x \to \infty$. (If u_1, u_2 are solutions, then $v = u_1 - u_2$ is harmonic in $\mathbb{R}^n$ and approaches 0 as $x \to \infty$; thus v is bounded and therefore constant, so $v = 0$.) If $n = 2$, then a similar argument shows that any solution of Poisson's equation such that $Du(x) \to 0$ as $|x| \to \infty$ differs from the Newtonian potential by a constant.

2.7.1. Second derivatives of the potential. In order to study the regularity of the Newtonian potential u in terms of f, we derive an integral representation for its second derivatives.

We write $\partial_i \partial_j = \partial_{ij}$, and let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

denote the Kronecker delta. In the following $\partial_i \Gamma(x-y)$ denotes the ith partial derivative of Γ evaluated at $x-y$, with similar notation for other derivatives. Thus,

$$\frac{\partial}{\partial y_i} \Gamma(x-y) = -\partial_i \Gamma(x-y).$$

THEOREM 2.26. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and $u = \Gamma * f$ where Γ is the Newtonian potential (2.12). If Ω is any smooth open set that contains the support of f, then*

$$\begin{aligned} \partial_{ij} u(x) = &\int_\Omega \partial_{ij}\Gamma(x-y)\big[f(y) - f(x)\big]\, dy \\ &- f(x) \int_{\partial\Omega} \partial_i \Gamma(x-y) \nu_j(y)\, dS(y). \end{aligned} \tag{2.25}$$

PROOF. As before, the result is straightforward to prove if $x \notin \operatorname{spt} f$. We choose $\Omega \supset \operatorname{spt} f$ such that $x \notin \overline{\Omega}$. Then Γ is smooth on $\overline{\Omega}$ so we may differentiate under the integral sign to get

$$\partial_{ij} u(x) = \int_\Omega \partial_{ij}\Gamma(x-y) f(y)\, dy.,$$

which is (2.25) with $f(x) = 0$.

If $x \in \operatorname{spt} f$, we follow a similar procedure to the one used in the proof of Theorem 2.25: We differentiate under the integral sign in the convolution $u = \Gamma * f$ on f, cut out a ball of radius r about the singularity in Γ, apply Greens' theorem, and let $r \to 0^+$.

In detail, define $\Omega_r(x)$ as in (2.18), where $\Omega \supset \operatorname{spt} f$ is a smooth open set. Since Γ is locally integrable, the Lebesgue dominated convergence theorem implies that

$$\partial_{ij} u(x) = \int_\Omega \Gamma(x-y) \partial_{ij} f(y)\, dy = \lim_{r \to 0^+} \int_{\Omega_r(x)} \Gamma(x-y) \partial_{ij} f(y)\, dy. \tag{2.26}$$

For $x \neq y$, we have the identity

$$\begin{aligned} &\Gamma(x-y)\partial_{ij} f(y) - \partial_{ij}\Gamma(x-y) f(y) \\ &\qquad = \frac{\partial}{\partial y_i}\left[\Gamma(x-y)\partial_j f(y)\right] + \frac{\partial}{\partial y_j}\left[\partial_i \Gamma(x-y) f(y)\right]. \end{aligned}$$

Thus, using Green's theorem, we get

$$\begin{aligned}\int_{\Omega_r(x)} \Gamma(x-y)\partial_{ij} f(y)\,dy = \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y) f(y)\,dy \\ - \int_{\partial B_r(x)} \left[\Gamma(x-y)\partial_j f(y)\nu_i(y) + \partial_i\Gamma(x-y) f(y)\nu_j(y)\right]\,dS(y).\end{aligned} \tag{2.27}$$

In (2.27), ν denotes the radially outward unit normal vector on $\partial B_r(x)$, which accounts for the minus sign of the surface integral; the integral over the boundary $\partial\Omega$ vanishes because f is identically zero there.

We cannot take the limit of the integral over $\Omega_r(x)$ directly, since $\partial_{ij}\Gamma$ is not locally integrable. To obtain a limiting integral that is convergent, we write

$$\begin{aligned}&\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y) f(y)\,dy \\ &\quad= \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy + f(x)\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\,dy \\ &\quad= \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy \\ &\qquad - f(x)\left[\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) - \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\right].\end{aligned}$$

Using this expression in (2.27) and using the result in (2.26), we get

$$\begin{aligned}\partial_{ij}u(x) = \lim_{r\to 0^+} \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy \\ - f(x)\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) \\ - \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\big[f(y)-f(x)\big]\nu_j(y)\,dS(y) \\ - \int_{\partial B_r(x)} \Gamma(x-y)\partial_j f(y)\nu_i(y)\,dS(y).\end{aligned} \tag{2.28}$$

Since f is smooth, the function $y \mapsto \partial_{ij}\Gamma(x-y)\,[f(y)-f(x)]$ is integrable on Ω, and by the Lebesgue dominated convergence theorem

$$\lim_{r\to 0^+} \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy = \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy.$$

We also have

$$\begin{aligned}\lim_{r\to 0^+} \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\big[f(y)-f(x)\big]\nu_j(y)\,dS(y) = 0, \\ \lim_{r\to 0^+} \int_{\partial B_r(x)} \Gamma(x-y)\partial_j f(y)\nu_i(y)\,dS(y) = 0.\end{aligned}$$

Using these limits in (2.28), we get (2.25). □

Note that if $\Omega' \supset \Omega \supset \operatorname{spt} f$, then writing

$$\Omega' = \Omega \cup (\Omega' \setminus \Omega)$$

and using the divergence theorem, we get

$$\begin{aligned}
&\int_{\Omega'} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy - f(x)\int_{\partial\Omega'} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\\
&\quad= \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy\\
&\qquad - f(x)\left[\int_{\partial\Omega'} \partial_i\Gamma(x-y)\nu_j(x-y)\,dS(y) + \int_{\Omega'\setminus\Omega} \partial_{ij}\Gamma(x-y)\,dy\right]\\
&\quad= \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy - f(x)\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y).
\end{aligned}$$

Thus, the expression on the right-hand side of (2.25) does not depend on Ω provided that it contains the support of f. In particular, we can choose Ω to be a sufficiently large ball centered at x.

Corollary 2.27. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and $u = \Gamma * f$ where Γ is the Newtonian potential (2.12). Then*

$$\partial_{ij}u(x) = \int_{B_R(x)} \partial_{ij}\Gamma(x-y)\left[f(y)-f(x)\right]\,dy - \frac{1}{n}f(x)\delta_{ij} \tag{2.29}$$

where $B_R(x)$ is any open ball centered at x that contains the support of f.

Proof. In (2.25), we choose $\Omega = B_R(x) \supset \operatorname{spt} f$. From (2.14), we have

$$\begin{aligned}
\int_{\partial B_R(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) &= \int_{\partial B_R(x)} \frac{-(x_i-y_i)}{n\alpha_n|x-y|^n}\,\frac{y_j-x_j}{|y-x|}\,dS(y)\\
&= \int_{\partial B_R(0)} \frac{y_iy_j}{n\alpha_n|y|^{n+1}}\,dS(y)
\end{aligned}$$

If $i \neq j$, then y_iy_j is odd under a reflection $y_i \mapsto -y_i$, so this integral is zero. If $i = j$, then the value of the integral does not depend on i, since we may transform the i-integral into an i'-integral by a rotation. Therefore

$$\begin{aligned}
\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{y_i^2}{|y|^{n+1}}\,dS(y) &= \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{y_i^2}{|y|^{n+1}}\,dS(y)\right)\\
&= \frac{1}{n}\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{1}{|y|^{n-1}}\,dS(y)\\
&= \frac{1}{n}.
\end{aligned}$$

It follows that

$$\int_{\partial B_R(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) = \frac{1}{n}\delta_{ij}.$$

Using this result in (2.25), we get (2.29). □

2.7.2. Hölder estimates. We want to derive estimates of the derivatives of the Newtonian potential $u = \Gamma * f$ in terms of the source density f. We continue to assume that $f \in C_c^\infty(\mathbb{R}^n)$; the estimates extend by a density argument to any Hölder-continuous function f with compact support (or sufficiently rapid decay at infinity).

In one space dimension, a solution of the ODE

$$-u'' = f$$

is given in terms of the potential (2.13) by

$$u(x) = -\frac{1}{2}\int |x-y|\, f(y)\, dy.$$

If $f \in C_c(\mathbb{R})$, then obviously $u \in C^2(\mathbb{R})$ and $\max |u''| = \max |f|$.

In more than one space dimension, however, it is not possible estimate the maximum norm of the second derivative D^2u of the potential $u = \Gamma * f$ in terms of the maximum norm of f, and there exist functions $f \in C_c(\mathbb{R}^n)$ for which $u \notin C^2(\mathbb{R}^n)$.

Nevertheless, if we measure derivatives in an appropriate way, we gain two derivatives in solving the Laplace equation (and other second-order elliptic PDEs). The fact that in inverting the Laplacian we gain as many derivatives as the order of the PDE is the essential point of elliptic regularity theory; this does not happen for many other types of PDEs, such as hyperbolic PDEs.

In particular, if we measure derivatives in terms of their Hölder continuity, we can estimate the $C^{2,\alpha}$-norm of u in terms of the $C^{0,\alpha}$-norm of f. These Hölder estimates were used by Schauder[3] to develop a general existence theory for elliptic PDEs with Hölder continuous coefficients, typically referred to as the Schauder theory [**10**].

Here, we will derive Hölder estimates for the Newtonian potential.

Theorem 2.28. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$ and $0 < \alpha < 1$. If $u = \Gamma * f$ where Γ is the Newtonian potential (2.12), then*

$$[\partial_{ij}u]_{0,\alpha} \le C\,[f]_{0,\alpha}$$

where $[\cdot]_{0,\alpha}$ denotes the Hölder semi-norm (1.1) and C is a constant that depends only on α and n.

Proof. Let Ω be a smooth open set that contains the support of f. We write (2.25) as

$$\partial_{ij}u = Tf - fg \tag{2.30}$$

where the linear operator

$$T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is defined by

$$Tf(x) = \int_\Omega K(x-y)\,[f(y) - f(x)]\; dy, \qquad K = \partial_{ij}\Gamma,$$

and the function $g : \mathbb{R}^n \to \mathbb{R}$ is given by

$$g(x) = \int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\, dS(y). \tag{2.31}$$

If $x, x' \in \mathbb{R}^n$, then

$$\partial_{ij}u(x) - \partial_{ij}u(x') = Tf(x) - Tf(x') - [f(x)g(x) - f(x')g(x')]$$

[3] Juliusz Schauder (1899–1943) was a Polish mathematician. In addition to the Schauder theory for elliptic PDEs, he is known for the Leray-Schauder fixed point theorem, and Schauder bases of a Banach space. He was killed by the Nazi's while they occupied Lvov during the second world war.

The main part of the proof is to estimate the difference of the terms that involve Tf.

In order to do this, let

$$\bar{x} = \frac{1}{2}(x + x'), \qquad \delta = |x - x'|,$$

and choose Ω so that it contains $B_{2\delta}(\bar{x})$. We have

$$\begin{aligned} &Tf(x) - Tf(x') \\ &\quad = \int_\Omega \{K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]\}\, dy. \end{aligned} \tag{2.32}$$

We will separate the the integral over Ω in (2.32) into two parts: (a) $|y - \bar{x}| < \delta$; (b) $|y - \bar{x}| \geq \delta$. In region (a), which contains the points $y = x$, $y = x'$ where K is singular, we will use the Hölder continuity of f and the smallness of the integration region to estimate the integral. In region (b), we will use the Hölder continuity of f and the smoothness of K to estimate the integral.

(a) Suppose that $|y - \bar{x}| < \delta$, meaning that $y \in B_\delta(\bar{x})$. Then

$$|x - y| \leq |x - \bar{x}| + |\bar{x} - y| \leq \frac{3}{2}\delta,$$

so $y \in B_{3\delta/2}(x)$, and similarly for x'. Using the Hölder continuity of f and the fact that K is homogeneous of degree $-n$, we have

$$\begin{aligned} &|K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]| \\ &\qquad\qquad \leq C[f]_{0,\alpha}\left\{|x-y|^{\alpha-n} + |x'-y|^{\alpha-n}\right\}. \end{aligned}$$

Thus, using C to denote a generic constant depending on α and n, we get

$$\begin{aligned} &\int_{B_\delta(\bar{x})} |K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]|\, dy \\ &\qquad \leq C[f]_{0,\alpha}\int_{B_\delta(\bar{x})}\left[|x-y|^{\alpha-n} + |x'-y|^{\alpha-n}\right] dy \\ &\qquad \leq C[f]_{0,\alpha}\int_{B_{3\delta/2}(0)} |y|^{\alpha-n} dy \\ &\qquad \leq C[f]_{0,\alpha}\,\delta^\alpha. \end{aligned}$$

(b) Suppose that $|y - \bar{x}| \geq \delta$. We write

$$\begin{aligned} &K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')] \\ &\quad = [K(x-y) - K(x'-y)][f(y) - f(x)] - K(x'-y)[f(x) - f(x')] \end{aligned} \tag{2.33}$$

and estimate the two terms on the right hand side separately. For the first term, we use the the Hölder continuity of f and the smoothness of K; for the second term we use the Hölder continuity of f and the divergence theorem to estimate the integral of K.

(b1) Since DK is homogeneous of degree $-(n+1)$, the mean value theorem implies that

$$|K(x-y) - K(x'-y)| \leq C\frac{|x - x'|}{|\xi - y|^{n+1}}$$

for $\xi = \theta x + (1-\theta)x'$ with $0 < \theta < 1$. Using this estimate and the Hölder continuity of f, we get

$$\left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right| \leq C\,[f]_{0,\alpha}\,\delta \frac{|y-x|^\alpha}{|\xi-y|^{n+1}}.$$

We have

$$\begin{aligned} |y-x| &\leq |y-\bar{x}| + |\bar{x}-x| = |y-\bar{x}| + \frac{1}{2}\delta \leq \frac{3}{2}|y-\bar{x}|, \\ |\xi-y| &\geq |y-\bar{x}| - |\bar{x}-\xi| \geq |y-\bar{x}| - \frac{1}{2}\delta \geq \frac{1}{2}|y-\bar{x}|. \end{aligned}$$

It follows that

$$\left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right| \leq C\,[f]_{0,\alpha}\,\delta|y-\bar{x}|^{\alpha-n-1}.$$

Thus,

$$\begin{aligned} &\int_{\Omega\setminus B_\delta(\bar{x})} \left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right|\,dy \\ &\qquad\qquad \leq \int_{\mathbb{R}^n\setminus B_\delta(\bar{x})} \left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right|\,dy \\ &\qquad\qquad \leq C\,[f]_{0,\alpha}\,\delta \int_{|y|\geq\delta} |y|^{\alpha-n-1}\,dy \\ &\qquad\qquad \leq C\,[f]_{0,\alpha}\,\delta^\alpha. \end{aligned}$$

Note that the integral does not converge at infinity if $\alpha = 1$; this is where we require $\alpha < 1$.

(b2) To estimate the second term in (2.33), we suppose that $\Omega = B_R(\bar{x})$ where $B_R(\bar{x})$ contains the support of f and $R \geq 2\delta$. (All of the estimates above apply for this choice of Ω.) Writing $K = \partial_{ij}\Gamma$ and using the divergence theorem we get

$$\begin{aligned} &\int_{B_R(\bar{x})\setminus B_\delta(\bar{x})} K(x-y)\,dy \\ &\qquad = \int_{\partial B_R(\bar{x})} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) - \int_{\partial B_\delta(\bar{x})} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y). \end{aligned}$$

If $y \in \partial B_R(\bar{x})$, then

$$|x-y| \geq |y-\bar{x}| - |\bar{x}-x| \geq R - \frac{1}{2}\delta \geq \frac{3}{4}R;$$

and If $y \in \partial B_\delta(\bar{x})$, then

$$|x-y| \geq |y-\bar{x}| - |\bar{x}-x| \geq \delta - \frac{1}{2}\delta \geq \frac{1}{2}\delta.$$

Thus, using the fact that $D\Gamma$ is homogeneous of degree $-n+1$, we compute that

$$\int_{\partial B_R(\bar{x})} \left|\partial_i\Gamma(x-y)\nu_j(y)\right|\,dS(y) \leq CR^{n-1}\frac{1}{R^{n-1}} \leq C \tag{2.34}$$

and

$$\int_{\partial B_\delta(\bar{x})} \left|\partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\right| C\delta^{n-1}\frac{1}{\delta^{n-1}} \leq C$$

Thus, using the Hölder continuity of f, we get

$$\left|[f(x) - f(x')] \int_{\Omega\setminus B_\delta(\bar{x})} K(x'-y)\,dy\right| \le C\,[f]_{0,\alpha}\,\delta^\alpha.$$

Putting these estimates together, we conclude that

$$|Tf(x) - Tf(x')| \le C\,[f]_{0,\alpha}\,|x-x'|^\alpha$$

where C is a constant that depends only on α and n.

(c) Finally, to estimate the Hölder norm of the remaining term fg in (2.30), we continue to assume that $\Omega = B_R(\bar{x})$. From (2.31),

$$g(\bar{x}+h) = \int_{\partial B_R(0)} \partial_i\Gamma(h-y)\nu_j(y)\,dS(y).$$

Changing $y \mapsto -y$ in the integral, we find that $g(\bar{x}+h) = g(\bar{x}-h)$. Hence $g(x) = g(x')$. Moreover, from (2.34), we have $|g(x)| \le C$. It therefore follows that

$$|f(x)g(x) - f(x')g(x')| \le C\,|f(x)-f(x')| \le C\,[f]_{0,\alpha}\,|x-x'|^\alpha,$$

which completes the proof. □

These Hölder estimates, and their generalizations, are fundamental to theory of elliptic PDEs. Their derivation by direct estimation of the Newtonian potential is only one of many methods to obtain them (although it was the original method). For example, they can also be obtained by the use of Campanato spaces, which provide Hölder estimates in terms of suitable integral norms [**12**], or by the use of Littlewood-Payley theory, which provides Hölder estimates in terms of dyadic decompositions of the Fourier transform [**2**].

2.8. Singular integral operators

Using (2.29), we may define a linear operator

$$T_{ij} : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

that gives the second derivatives of a function in terms of its Laplacian,

$$\partial_{ij}u = T_{ij}\Delta u.$$

Explicitly,

$$T_{ij}f(x) = \int_{B_R(x)} K_{ij}(x-y)\,[f(y)-f(x)]\;dy + \frac{1}{n}f(x)\delta_{ij} \tag{2.35}$$

where $B_R(x) \supset \operatorname{spt} f$ and $K_{ij} = -\partial_{ij}\Gamma$ is given by

$$K_{ij}(x) = \frac{1}{\alpha_n|x|^n}\left(\frac{1}{n}\delta_{ij} - \frac{x_ix_j}{|x|^2}\right). \tag{2.36}$$

This function is homogeneous of degree $-n$, the borderline power for integrability, so it is not locally integrable. Thus, Young's inequality does not imply that convolution with K_{ij} is a bounded operator on L^∞_{loc}, which explains why we cannot bound the maximum norm of D^2u in terms of the maximum norm of f.

The kernel K_{ij} in (2.36) has zero integral over any sphere, meaning that

$$\int_{B_R(0)} K_{ij}(y)\,dS(y) = 0.$$

Thus, we may alternatively write T_{ij} as

$$\begin{aligned} T_{ij}f(x) - \frac{1}{n}f(x)\delta_{ij} &= \lim_{\epsilon\to 0^+}\int_{B_R(x)\setminus B_\epsilon(x)} K_{ij}(x-y)\left[f(y)-f(x)\right]\,dy \\ &= \lim_{\epsilon\to 0^+}\int_{B_R(x)\setminus B_\epsilon(x)} K_{ij}(x-y)f(y)\,dy \\ &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} K_{ij}(x-y)f(y)\,dy. \end{aligned}$$

This is an example of a singular integral operator.

The operator T_{ij} can also be expressed in terms of the Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n}\int f(x)e^{-i\cdot\xi}\,dx$$

as

$$\widehat{(T_{ij}f)}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}\hat{f}(\xi).$$

Since the multiplier $m_{ij} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$m_{ij}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}$$

belongs to $L^\infty(\mathbb{R}^n)$, it follows from Plancherel's theorem that T_{ij} extends to a bounded linear operator on $L^2(\mathbb{R}^n)$.

In more generality, consider a function $K : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable in $\mathbb{R}^n \setminus 0$ and satisfies the following conditions:

$$\begin{aligned} K(\lambda x) &= \frac{1}{\lambda^n}K(x) \qquad \text{for } \lambda > 0; \\ \int_{\partial B_R(0)} K\,dS &= 0 \qquad \text{for } R > 0. \end{aligned} \tag{2.37}$$

That is, K is homogeneous of degree $-n$, and its integral over any sphere centered at zero is zero. We may then write

$$K(x) = \frac{\Omega\left(\hat{x}\right)}{|x|^n}, \qquad \hat{x} = \frac{x}{|x|}$$

where $\Omega : \mathbb{S}^{n-1} \to \mathbb{R}$ is a C^1-function such that

$$\int_{\mathbb{S}^{n-1}} \Omega\,dS = 0.$$

We define a singular integral operator $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ of convolution type with smooth, homogeneous kernel K by

$$Tf(x) = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} K(x-y)f(y)\,dy. \tag{2.38}$$

This operator is well-defined, since if $B_R(x) \supset \operatorname{spt} f$, we may write

$$\begin{aligned}
Tf(x) &= \lim_{\epsilon \to 0^+} \int_{B_R(x)\setminus B_\epsilon(x)} K(x-y) f(y)\, dy. \\
&= \lim_{\epsilon \to 0^+} \left\{ \int_{B_R(x)\setminus B_\epsilon(x)} K(x-y)\left[f(y) - f(x)\right]\, dy \right. \\
&\qquad\qquad \left. + f(x) \int_{B_R(x)\setminus B_\epsilon(x)} K(x-y)\, dy \right\} \\
&= \int_{B_R(x)} K(x-y)\left[f(y) - f(x)\right]\, dy.
\end{aligned}$$

Here, we use the dominated convergence theorem and the fact that

$$\int_{B_R(0)\setminus B_\epsilon(0)} K(y)\, dy = 0$$

since K has zero mean over spheres centered at the origin. Thus, the cancelation due to the fact that K has zero mean over spheres compensates for the non-integrability of K at the origin to give a finite limit.

Calderón and Zygmund (1952) proved that such operators, and generalizations of them, extend to bounded linear operators on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$ (see e.g. [**3**]). As a result, we also 'gain' two derivatives in inverting the Laplacian when derivatives are measured in L^p for $1 < p < \infty$.