CHAPTER 4

Elliptic PDEs

One of the main advantages of extending the class of solutions of a PDE from classical solutions with continuous derivatives to weak solutions with weak derivatives is that it is easier to prove the existence of weak solutions. Having established the existence of weak solutions, one may then study their properties, such as uniqueness and regularity, and perhaps prove under appropriate assumptions that the weak solutions are, in fact, classical solutions.

There is often considerable freedom in how one defines a weak solution of a PDE; for example, the function space to which a solution is required to belong is not given *a priori* by the PDE itself. Typically, we look for a weak formulation that reduces to the classical formulation under appropriate smoothness assumptions and which is amenable to a mathematical analysis; the notion of solution and the spaces to which solutions belong are dictated by the available estimates and analysis.

4.1. Weak formulation of the Dirichlet problem

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain Ω in $\mathbb{R}^n$,

$$-\Delta u = f \qquad \text{in } \Omega, \tag{4.1}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4.2}$$

First, suppose that the boundary of Ω is smooth and $u, f : \overline{\Omega} \to \mathbb{R}$ are smooth functions. Multiplying (4.1) by a test function ϕ, integrating the result over Ω, and using the divergence theorem, we get

$$\int_\Omega Du \cdot D\phi \, dx = \int_\Omega f\phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{4.3}$$

The boundary terms vanish because $\phi = 0$ on the boundary. Conversely, if f and Ω are smooth, then any smooth function u that satisfies (4.3) is a solution of (4.1).

Next, we formulate weaker assumptions under which (4.3) makes sense. We use the flexibility of choice to define weak solutions with L^2-derivatives that belong to a Hilbert space; this is helpful because Hilbert spaces are easier to work with than Banach spaces.[1] It also leads to a variational form of the equation that is symmetric in the solution u and the test function ϕ.

By the Cauchy-Schwartz inequality, the integral on the left-hand side of (4.3) is finite if Du belongs to $L^2(\Omega)$, so we suppose that $u \in H^1(\Omega)$. We impose the boundary condition (4.2) in a weak sense by requiring that $u \in H_0^1(\Omega)$. The left hand side of (4.3) then extends by continuity to $\phi \in H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}$.

[1] We would need to use Banach spaces to study the solutions of Laplace's equation whose derivatives lie in L^p for $p \neq 2$, and we may be forced to use Banach spaces for some PDEs, especially if they are nonlinear.

The right hand side of (4.3) is well-defined for all $\phi \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$, but this is not the most general f for which it makes sense; we can define the right-hand for any f in the dual space of $H_0^1(\Omega)$.

Definition 4.1. The space of bounded linear maps $f : H_0^1(\Omega) \to \mathbb{R}$ is denoted by $H^{-1}(\Omega) = H_0^1(\Omega)^*$, and the action of $f \in H^{-1}(\Omega)$ on $\phi \in H_0^1(\Omega)$ by $\langle f, \phi \rangle$. The norm of $f \in H^{-1}(\Omega)$ is given by

$$\|f\|_{H^{-1}} = \sup\left\{ \frac{|\langle f, \phi\rangle|}{\|\phi\|_{H_0^1}} : \phi \in H_0^1, \phi \neq 0 \right\}.$$

A function $f \in L^2(\Omega)$ defines a linear functional $F_f \in H^{-1}(\Omega)$ by

$$\langle F_f, v\rangle = \int_\Omega f v \, dx = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega).$$

Here, $(\cdot,\cdot)_{L^2}$ denotes the standard inner product on $L^2(\Omega)$. The functional F_f is bounded on $H_0^1(\Omega)$ with $\|F_f\|_{H^{-1}} \le \|f\|_{L^2}$ since, by the Cauchy-Schwartz inequality,

$$|\langle F_f, v\rangle| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_{H_0^1}.$$

We identify F_f with f, and write both simply as f.

Such linear functionals are, however, not the only elements of $H^{-1}(\Omega)$. As we will show below, $H^{-1}(\Omega)$ may be identified with the space of distributions on Ω that are sums of first-order distributional derivatives of functions in $L^2(\Omega)$.

Thus, after identifying functions with regular distributions, we have the following triple of Hilbert spaces

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \qquad H^{-1}(\Omega) = H_0^1(\Omega)^*.$$

Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, then

$$\langle f, u\rangle = (f, u)_{L^2},$$

so the duality pairing coincides with the L^2-inner product when both are defined.

This discussion motivates the following definition.

Definition 4.2. Let Ω be an open set in $\mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is a weak solution of (4.1)–(4.2) if: (a) $u \in H_0^1(\Omega)$; (b)

$$\int_\Omega Du \cdot D\phi \, dx = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega). \tag{4.4}$$

Here, strictly speaking, 'function' means an equivalence class of functions with respect to pointwise a.e. equality.

We have assumed homogeneous boundary conditions to simplify the discussion. If Ω is smooth and $g : \partial\Omega \to \mathbb{R}$ is a function on the boundary that is in the range of the trace map $T : H^1(\Omega) \to L^2(\partial\Omega)$, say $g = Tw$, then we obtain a weak formulation of the nonhomogeneous Dirichet problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

by replacing (a) in Definition 4.2 with the condition that $u - w \in H_0^1(\Omega)$. The definition is otherwise the same. The range of the trace map on $H^1(\Omega)$ for a smooth domain Ω is the fractional-order Sobolev space $H^{1/2}(\partial\Omega)$; thus if the boundary data g is so rough that $g \notin H^{1/2}(\partial\Omega)$, then there is no solution $u \in H^1(\Omega)$ of the nonhomogeneous BVP.

4.2. Variational formulation

Definition 4.2 of a weak solution in is closely connected with the variational formulation of the Dirichlet problem for Poisson's equation. To explain this connection, we first summarize some definitions of the differentiability of functionals (scalar-valued functions) acting on a Banach space.

DEFINITION 4.3. A functional $J : X \to \mathbb{R}$ on a Banach space X is differentiable at $x \in X$ if there is a bounded linear functional $A : X \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|J(x+h) - J(x) - Ah|}{\|h\|_X} = 0.$$

If A exists, then it is unique, and it is called the derivative, or differential, of J at x, denoted $DJ(x) = A$.

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If J is differentiable at every point of X, then $DJ : X \to X^*$ maps $x \in X$ to the linear functional $DJ(x) \in X^*$ that approximates J near x.

A weaker notion of differentiability (even for functions $J : \mathbb{R}^2 \to \mathbb{R}$ — see Example 4.4) is the existence of directional derivatives

$$\delta J(x; h) = \lim_{\epsilon\to 0}\left[\frac{J(x+\epsilon h) - J(x)}{\epsilon}\right] = \frac{d}{d\epsilon} J(x+\epsilon h)\bigg|_{\epsilon=0}.$$

If the directional derivative at x exists for every $h \in X$ and is a bounded linear functional on h, then $\delta J(x; h) = \delta J(x)h$ where $\delta J(x) \in X^*$. We call $\delta J(x)$ the Gâteaux derivative of J at x. The derivative DJ is then called the Fréchet derivative to distinguish it from the directional or Gâteaux derivative. If J is differentiable at x, then it is Gâteaux-differentiable at x and $DJ(x) = \delta J(x)$, but the converse is not true.

EXAMPLE 4.4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0, 0) = 0$ and

$$f(x, y) = \left(\frac{xy^2}{x^2 + y^4}\right)^2 \qquad \text{if } (x, y) \neq (0, 0).$$

Then f is Gâteaux-differentiable at 0, with $\delta f(0) = 0$, but f is not Fréchet-differentiable at 0.

If $J : X \to \mathbb{R}$ attains a local minimum at $x \in X$ and J is differentiable at x, then for every $h \in X$ the function $J_{x;h} : \mathbb{R} \to \mathbb{R}$ defined by $J_{x;h}(t) = J(x + th)$ is differentiable at $t = 0$ and attains a minimum at $t = 0$. It follows that

$$\frac{dJ_{x;h}}{dt}(0) = \delta J(x; h) = 0 \qquad \text{for every } h \in X.$$

Hence $DJ(x) = 0$. Thus, just as in multivariable calculus, an extreme point of a differentiable functional is a critical point where the derivative is zero.

Given $f \in H^{-1}(\Omega)$, define a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |Du|^2 \, dx - \langle f, u\rangle. \tag{4.5}$$

Clearly, J is well-defined.

PROPOSITION 4.5. *The functional $J : H_0^1(\Omega) \to \mathbb{R}$ in (4.5) is differentiable. Its derivative $DJ(u) : H_0^1(\Omega) \to \mathbb{R}$ at $u \in H_0^1(\Omega)$ is given by*

$$DJ(u)h = \int_\Omega Du \cdot Dh \, dx - \langle f, h\rangle \qquad \text{for } h \in H_0^1(\Omega).$$

PROOF. Given $u \in H_0^1(\Omega)$, define the linear map $A : H_0^1(\Omega) \to \mathbb{R}$ by

$$Ah = \int_\Omega Du \cdot Dh \, dx - \langle f, h\rangle.$$

Then A is bounded, with $\|A\| \leq \|Du\|_{L^2} + \|f\|_{H^{-1}}$, since

$$|Ah| \leq \|Du\|_{L^2}\|Dh\|_{L^2} + \|f\|_{H^{-1}}\|h\|_{H_0^1} \leq \left(\|Du\|_{L^2} + \|f\|_{H^{-1}}\right)\|h\|_{H_0^1}.$$

For $h \in H_0^1(\Omega)$, we have

$$J(u+h) - J(u) - Ah = \frac{1}{2}\int_\Omega |Dh|^2 \, dx.$$

It follows that

$$|J(u+h) - J(u) - Ah| \leq \frac{1}{2}\|h\|_{H_0^1}^2,$$

and therefore

$$\lim_{h\to 0} \frac{|J(u+h) - J(u) - Ah|}{\|h\|_{H_0^1}} = 0,$$

which proves that J is differentiable on $H_0^1(\Omega)$ with $DJ(u) = A$. □

Note that $DJ(u) = 0$ if and only if u is a weak solution of Poisson's equation in the sense of Definition 4.2. Thus, we have the following result.

COROLLARY 4.6. *If $J : H_0^1(\Omega) \to \mathbb{R}$ defined in (4.5) attains a minimum at $u \in H_0^1(\Omega)$, then u is a weak solution of $-\Delta u = f$ in the sense of Definition 4.2.*

In the direct method of the calculus of variations, we prove the existence of a minimizer of J by showing that a minimizing sequence $\{u_n\}$ converges in a suitable sense to a minimizer u. This minimizer is then a weak solution of (4.1)–(4.2). We will not follow this method here, and instead establish the existence of a weak solution by use of the Riesz representation theorem. The Riesz representation theorem is, however, typically proved by a similar argument to the one used in the direct method of the calculus of variations, so in essence the proofs are equivalent.

4.3. The space $H^{-1}(\Omega)$

The negative order Sobolev space $H^{-1}(\Omega)$ can be described as a space of distributions on Ω.

THEOREM 4.7. *The space $H^{-1}(\Omega)$ consists of all distributions $f \in \mathcal{D}'(\Omega)$ of the form*

$$f = f_0 + \sum_{i=1}^n \partial_i f_i \qquad \text{where } f_0, f_i \in L^2(\Omega). \tag{4.6}$$

These distributions extend uniquely by continuity from $\mathcal{D}(\Omega)$ to bounded linear functionals on $H_0^1(\Omega)$. Moreover,

$$\|f\|_{H^{-1}(\Omega)} = \inf\left\{\left(\sum_{i=0}^n \int_\Omega f_i^2 \, dx\right)^{1/2} : \text{such that } f_0,\ f_i \text{ satisfy (4.6)}\right\}. \tag{4.7}$$

Proof. First suppose that $f \in H^{-1}(\Omega)$. By the Riesz representation theorem there is a function $g \in H_0^1(\Omega)$ such that

$$\langle f, \phi\rangle = (g, \phi)_{H_0^1} \qquad \text{for all } \phi \in H_0^1(\Omega). \tag{4.8}$$

Here, $(\cdot,\cdot)_{H_0^1}$ denotes the standard inner product on $H_0^1(\Omega)$,

$$(u, v)_{H_0^1} = \int_\Omega (uv + Du \cdot Dv)\, dx.$$

Identifying a function $g \in L^2(\Omega)$ with its corresponding regular distribution, restricting f to $\phi \in \mathcal{D}(\Omega) \subset H_0^1(\Omega)$, and using the definition of the distributional derivative, we have

$$\begin{aligned}
\langle f, \phi\rangle &= \int_\Omega g\phi\, dx + \sum_{i=1}^n \int_\Omega \partial_i g\, \partial_i \phi\, dx \\
&= \langle g, \phi\rangle + \sum_{i=1}^n \langle \partial_i g, \partial_i \phi\rangle \\
&= \left\langle g - \sum_{i=1}^n \partial_i g_i, \phi \right\rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega),
\end{aligned}$$

where $g_i = \partial_i g \in L^2(\Omega)$. Thus the restriction of every $f \in H^{-1}(\Omega)$ from $H_0^1(\Omega)$ to $\mathcal{D}(\Omega)$ is a distribution

$$f = g - \sum_{i=1}^n \partial_i g_i$$

of the form (4.6). Also note that taking $\phi = g$ in (4.8), we get $\langle f, g\rangle = \|g\|^2_{H_0^1}$, which implies that

$$\|f\|_{H^{-1}} \geq \|g\|_{H_0^1} = \left(\int_\Omega g^2\, dx + \sum_{i=1}^n \int_\Omega g_i^2\, dx \right)^{1/2},$$

which proves inequality in one direction of (4.7).

Conversely, suppose that $f \in \mathcal{D}'(\Omega)$ is a distribution of the form (4.6). Then, using the definition of the distributional derivative, we have for any $\phi \in \mathcal{D}(\Omega)$ that

$$\langle f, \phi\rangle = \langle f_0, \phi\rangle + \sum_{i=1}^n \langle \partial_i f_i, \phi\rangle = \langle f_0, \phi\rangle - \sum_{i=1}^n \langle f_i, \partial_i \phi\rangle.$$

Use of the Cauchy-Schwartz inequality gives

$$|\langle f, \phi\rangle| \leq \left(\langle f_0, \phi\rangle^2 + \sum_{i=1}^n \langle f_i, \partial_i \phi\rangle^2 \right)^{1/2}.$$

Moreover, since the f_i are regular distributions belonging to $L^2(\Omega)$

$$|\langle f_i, \partial_i \phi\rangle| = \left| \int_\Omega f_i \partial_i \phi\, dx \right| \leq \left(\int_\Omega f_i^2\, dx \right)^{1/2} \left(\int_\Omega \partial_i \phi^2\, dx \right)^{1/2},$$

so

$$|\langle f, \phi\rangle| \leq \left[\left(\int_\Omega f_0^2\, dx \right) \left(\int_\Omega \phi^2\, dx \right) + \sum_{i=1}^n \left(\int_\Omega f_i^2\, dx \right) \left(\int_\Omega \partial_i \phi^2\, dx \right) \right]^{1/2},$$

and

$$
\begin{aligned}
|\langle f, \phi\rangle| &\leq \left(\int_\Omega f_0^2\,dx + \sum_{i=1}^n \int_\Omega f_i^2\,dx\right)^{1/2} \left(\int_\Omega \phi^2 + \int_\Omega \partial_i \phi^2\,dx\right)^{1/2} \\
&\leq \left(\sum_{i=0}^n \int_\Omega f_i^2\,dx\right)^{1/2} \|\phi\|_{H_0^1}
\end{aligned}
$$

Thus the distribution $f : \mathcal{D}(\Omega) \to \mathbb{R}$ is bounded with respect to the $H_0^1(\Omega)$-norm on the dense subset $\mathcal{D}(\Omega)$. It therefore extends in a unique way to a bounded linear functional on $H_0^1(\Omega)$, which we still denote by f. Moreover,

$$
\|f\|_{H^{-1}} \leq \left(\sum_{i=0}^n \int_\Omega f_i^2\,dx\right)^{1/2},
$$

which proves inequality in the other direction of (4.7). □

The dual space of $H^1(\Omega)$ cannot be identified with a space of distributions on Ω because $\mathcal{D}(\Omega)$ is not a dense subspace. Any linear functional $f \in H^1(\Omega)^*$ defines a distribution by restriction to $\mathcal{D}(\Omega)$, but the same distribution arises from different linear functionals. Conversely, any distribution $T \in \mathcal{D}'(\Omega)$ that is bounded with respect to the H^1-norm extends uniquely to a bounded linear functional on H_0^1, but the extension of the functional to the orthogonal complement $(H_0^1)^\perp$ in H^1 is arbitrary (subject to maintaining its boundedness). Roughly speaking, distributions are defined on functions whose boundary values or trace is zero, but general linear functionals on H^1 depend on the trace of the function on the boundary $\partial\Omega$.

Example 4.8. The one-dimensional Sobolev space $H^1(0,1)$ is imbedded in the space $C([0,1])$ of continuous functions, since $p > n$ for $p = 2$ and $n = 1$. In fact, according to the Sobolev imbedding theorem $H^1(0,1) \hookrightarrow C^{0,1/2}([0,1])$, as can be seen directly from the Cauchy-Schwartz inequality:

$$
\begin{aligned}
|f(x) - f(y)| &\leq \int_y^x |f'(t)|\,dt \\
&\leq \left(\int_y^x 1\,dt\right)^{1/2} \left(\int_y^x |f'(t)|^2\,dt\right)^{1/2} \\
&\leq \left(\int_0^1 |f'(t)|^2\,dt\right)^{1/2} |x-y|^{1/2}.
\end{aligned}
$$

As usual, we identify an element of $H^1(0,1)$ with its continuous representative in $C([0,1])$. By the trace theorem,

$$
H_0^1(0,1) = \left\{u \in H^1(0,1) : u(0) = u(1)\right\}.
$$

The orthogonal complement is

$$
H_0^1(0,1)^\perp = \left\{u \in H^1(0,1) : \text{such that } (u,v)_{H^1} = 0 \text{ for every } v \in H_0^1(0,1)\right\}.
$$

This condition implies that $u \in H_0^1(0,1)^\perp$ if and only if

$$
\int_0^1 (uv + u'v')\,dx = 0 \qquad \text{for all } v \in H_0^1(0,1),
$$

which means that u is a weak solution of the ODE

$$-u'' + u = 0.$$

It follows that $u(x) = c_1 e^x + c_2 e^{-x}$, so

$$H^1(0,1) = H_0^1(0,1) \oplus E$$

where E is the two dimensional subspace of $H^1(0,1)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$. Thus,

$$H^1(0,1)^* = H^{-1}(0,1) \oplus E^*.$$

If $f \in H^1(0,1)^*$ and $u = u_0 + c_1 e^x + c_2 e^{-x}$ where $u_0 \in H_0^1(0,1)$, then

$$\langle f, u\rangle = \langle f_0, u_0\rangle + a_1 c_1 + a_2 c_2$$

where $f_0 \in H^{-1}(0,1)$ is the restriction of f to $H_0^1(0,1)$ and

$$a_1 = \langle f, e^x\rangle, \qquad a_2 = \langle f, e^{-x}\rangle.$$

The constants a_1, a_2 determine how the functional $f \in H^1(0,1)^*$ acts on the boundary values $u(0)$, $u(1)$ of a function $u \in H^1(0,1)$.

4.4. The Poincaré inequality for $H_0^1(\Omega)$

We cannot, in general, estimate a norm of a function in terms of a norm of its derivative since constant functions have zero derivative. Such estimates are possible if we add an additional condition that eliminates non-zero constant functions. For example, we can require that the function vanishes on the boundary of a domain, or that it has zero mean. We typically also need some sort of boundedness condition on the domain of the function, since even if a function vanishes at some point we cannot expect to estimate the size of a function over arbitrarily large distances by the size of its derivative. The resulting inequalities are called Poincaré inequalities.

The inequality we prove here is a basic example of a Poincaré inequality. We say that an open set Ω in $\mathbb{R}^n$ is bounded in some direction if there is a unit vector $e \in \mathbb{R}^n$ and constants a, b such that $a < x \cdot e < b$ for all $x \in \Omega$.

THEOREM 4.9. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded is some direction. Then there is a constant C such that*

$$\int_\Omega u^2\, dx \le C \int_\Omega |Du|^2\, dx \qquad \text{for all } u \in H_0^1(\Omega). \tag{4.9}$$

PROOF. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove the inequality for $u \in C_c^\infty(\Omega)$. The inequality is invariant under rotations and translations, so we can assume without loss of generality that the domain is bounded in the x_n-direction and lies between $0 < x_n < a$.

Writing $x = (x', x_n)$ where $x' = (x_1, \dots, x_{n-1})$, we have

$$|u(x', x_n)| = \left|\int_0^{x_n} \partial_n u(x', t)\, dt\right| \le \int_0^a |\partial_n u(x', t)|\, dt.$$

The Cauchy-Schwartz inequality implies that

$$\int_0^a |\partial_n u(x', t)|\, dt = \int_0^a 1 \cdot |\partial_n u(x', t)|\, dt \le a^{1/2}\left(\int_0^a |\partial_n u(x', t)|^2\, dt\right)^{1/2}.$$

Hence,

$$|u(x', x_n)|^2 \leq a \int_0^a |\partial_n u(x', t)|^2 \, dt.$$

Integrating this inequality with respect to x_n, we get

$$\int_0^a |u(x', x_n)|^2 \, dx_n \leq a^2 \int_0^a |\partial_n u(x', t)|^2 \, dt.$$

A further integration with respect to x' gives

$$\int_\Omega |u(x)|^2 \, dx \leq a^2 \int_\Omega |\partial_n u(x)|^2 \, dx.$$

Since $|\partial_n u| \leq |Du|$, the result follows with $C = a^2$. □

This inequality implies that we may use as an equivalent inner-product on H_0^1 an expression that involves only the derivatives of the functions and not the functions themselves.

Corollary 4.10. *If Ω is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product*

$$(u, v)_0 = \int_\Omega Du \cdot Dv \, dx \tag{4.10}$$

is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

Proof. We denote the norm associated with the inner-product (4.10) by

$$\|u\|_0 = \left(\int_\Omega |Du|^2 \, dx \right)^{1/2},$$

and the standard norm and inner product by

$$\begin{aligned} \|u\|_1 &= \left(\int_\Omega \left[u^2 + |Du|^2 \right] dx \right)^{1/2}, \\ (u, v)_1 &= \int_\Omega (uv + Du \cdot Dv) \, dx. \end{aligned} \tag{4.11}$$

Then, using the Poincaré inequality (4.9), we have

$$\|u\|_0 \leq \|u\|_1 \leq (C + 1)^{1/2} \|u\|_0.$$

Thus, the two norms are equivalent; in particular, $(H_0^1, (\cdot, \cdot)_0)$ is complete since $(H_0^1, (\cdot, \cdot)_1)$ is complete, so it is a Hilbert space with respect to the inner product (4.10). □

4.5. Existence of weak solutions of the Dirichlet problem

With these preparations, the existence of weak solutions is an immediate consequence of the Riesz representation theorem.

Theorem 4.11. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in the sense of Definition 4.2.*

PROOF. We equip $H_0^1(\Omega)$ with the inner product (4.10). Then, since Ω is bounded in some direction, the resulting norm is equivalent to the standard norm, and f is a bounded linear functional on $\left(H_0^1(\Omega), (,)_0\right)$. By the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, \phi)_0 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega),$$

which is equivalent to the condition that u is a weak solution. □

The same approach works for other symmetric linear elliptic PDEs. Let us give some examples.

EXAMPLE 4.12. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Then $u \in H_0^1(\Omega)$ is a weak solution if

$$\int_\Omega (Du \cdot D\phi + u\phi)\, dx = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

where $(\cdot, \cdot)_1$ is the standard inner product on $H_0^1(\Omega)$ given in (4.11). Thus, the Riesz representation theorem implies the existence of a unique weak solution.

Note that in this example and the next, we do not use the Poincaré inequality, so the result applies to arbitrary open sets, including $\Omega = \mathbb{R}^n$. In that case, $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, and we get a unique solution $u \in H^1(\mathbb{R}^n)$ of $-\Delta u + u = f$ for every $f \in H^{-1}(\mathbb{R}^n)$. Moreover, using the standard norms, we have $\|u\|_{H^1} = \|f\|_{H^{-1}}$. Thus the operator $-\Delta + I$ is an isometry of $H^1(\mathbb{R}^n)$ onto $H^{-1}(\mathbb{R}^n)$.

EXAMPLE 4.13. As a slight generalization of the previous example, suppose that $\mu > 0$. A function $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{aligned} -\Delta u + \mu u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.12}$$

if $(u, \phi)_\mu = \langle f, \phi \rangle$ for all $\phi \in H_0^1(\Omega)$ where

$$(u, v)_\mu = \int_\Omega (\mu uv + Du \cdot Dv)\, dx$$

The norm $\|\cdot\|_\mu$ associated with this inner product is equivalent to the standard one, since

$$\frac{1}{C}\|u\|_\mu^2 \le \|u\|_1^2 \le C\|u\|_\mu^2$$

where $C = \max\{\mu, 1/\mu\}$. We therefore again get the existence of a unique weak solution from the Riesz representation theorem.

EXAMPLE 4.14. Consider the last example for $\mu < 0$. If we have a Poincaré inequality $\|u\|_{L^2} \le C\|Du\|_L^2$ for Ω, which is the case if Ω is bounded in some direction, then

$$(u, u)_\mu = \int_\Omega \left(\mu u^2 + Du \cdot Dv\right)\, dx \ge (1 - C|\mu|) \int_\Omega |Du|^2\, dx.$$

Thus $\|u\|_\mu$ defines a norm on $H_0^1(\Omega)$ that is equivalent to the standard norm if $-1/C < \mu < 0$, and we get a unique weak solution in this case also.

For bounded domains, the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ such that there exists a nonzero solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda_n u$. The best constant in the Poincaré inequality can be shown to be the minimum eigenvalue λ_1, and this method does not work if $\mu \leq -\lambda_1$. For $\mu = -\lambda_n$, a weak solution of (4.12) does not exist for every $f \in H^{-1}(\Omega)$, and if one does exist it is not unique since we can add to it an arbitrary eigenfunction. Thus, not only does the method fail, but the conclusion of Theorem 4.11 may be false.

EXAMPLE 4.15. Consider the second order PDE

$$\begin{aligned} -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{4.13}$$

where the coefficient functions $a_{ij} : \Omega \to \mathbb{R}$ are symmetric ($a_{ij} = a_{ji}$), bounded, and satisfy the uniform ellipticity condition that for some $\theta > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n.$$

Also, assume that Ω is bounded in some direction. Then a weak formulation of (4.13) is that $u \in H_0^1(\Omega)$ and

$$a(u, \phi) = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega),$$

where the symmetric bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$a(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_i u \partial_j v \, dx.$$

The boundedness of a_{ij}, the uniform ellipticity condition, and the Poincaré inequality imply that a defines an inner product on H_0^1 which is equivalent to the standard one. An application of the Riesz representation theorem for the bounded linear functionals f on the Hilbert space (H_0^1, a) then implies the existence of a unique weak solution. We discuss a generalization of this example in greater detail in the next section.

4.6. General linear, second order elliptic PDEs

Consider PDEs of the form

$$Lu = f$$

where L is a linear differential operator of the form

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu, \tag{4.14}$$

acting on functions $u : \Omega \to \mathbb{R}$ where Ω is an open set in $\mathbb{R}^n$. A physical interpretation of such PDEs is described briefly in Section 4.A.

We assume that the given coefficients functions $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ satisfy

$$a_{ij}, b_i, c \in L^\infty(\Omega), \qquad a_{ij} = a_{ji}. \tag{4.15}$$

The operator L is elliptic if the matrix (a_{ij}) is positive definite. We will assume the stronger condition of uniformly ellipticity given in the next definition.

DEFINITION 4.16. The operator L in (4.14) is uniformly elliptic on Ω if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{4.16}$$

for x almost everywhere in Ω and every $\xi \in \mathbb{R}^n$.

This uniform ellipticity condition allows us to estimate the integral of $|Du|^2$ in terms of the integral of $\sum a_{ij}\partial_i u \partial_j u$.

EXAMPLE 4.17. The Laplacian operator $L = -\Delta$ is uniformly elliptic on any open set, with $\theta = 1$.

EXAMPLE 4.18. The Tricomi operator

$$L = y\partial_x^2 + \partial_y^2$$

is elliptic in $y > 0$ and hyperbolic in $y < 0$. For any $0 < \epsilon < 1$, L is uniformly elliptic in the strip $\{(x, y) : \epsilon < y < 1\}$, with $\theta = \epsilon$, but it is not uniformly elliptic in $\{(x, y) : 0 < y < 1\}$.

For $\mu \in \mathbb{R}$, we consider the Dirichlet problem for $L + \mu I$,

$$\begin{aligned} Lu + \mu u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.17}$$

We motivate the definition of a weak solution of (4.17) in a similar way to the motivation for the Laplacian: multiply the PDE by a test function $\phi \in C_c^\infty(\Omega)$, integrate over Ω, and use integration by parts, assuming that all functions and the domain are smooth. Note that

$$\int_\Omega \partial_i(b_i u)\phi\, dx = -\int_\Omega b_i u \partial_i \phi\, dx.$$

This leads to the condition that $u \in H_0^1(\Omega)$ is a weak solution of (4.17) with L given by (4.14) if

$$\int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j \phi - \sum_{i=1}^{n} b_i u \partial_i \phi + cu\phi \right\} dx + \mu \int_\Omega u\phi\, dx = \langle f, \phi \rangle$$

for all $\phi \in H_0^1(\Omega)$.

To write this condition more concisely, we define a bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j v - \sum_{i}^{n} b_i u \partial_i v + cuv \right\} dx. \tag{4.18}$$

This form is well-defined and bounded on $H_0^1(\Omega)$, as we check explicitly below. We denote the L^2-inner product by

$$(u, v)_{L^2} = \int_\Omega uv\, dx.$$

DEFINITION 4.19. Suppose that Ω is an open set in $\mathbb{R}^n$, $f \in H^{-1}(\Omega)$, and L is a differential operator (4.14) whose coefficients satisfy (4.15). Then $u : \Omega \to \mathbb{R}$ is a weak solution of (4.17) if: (a) $u \in H_0^1(\Omega)$; (b)

$$a(u, \phi) + \mu(u, \phi)_{L^2} = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

The form a in (4.18) is not symmetric unless $b_i = 0$. We have

$$a(v, u) = a^*(u, v)$$

where

$$a^*(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v + \sum_i^n b_i (\partial_i u) v + cuv \right\} dx \tag{4.19}$$

is the bilinear form associated with the formal adjoint L^* of L,

$$L^* u = - \sum_{i,j=1}^n \partial_i \left(a_{ij} \partial_j u \right) - \sum_{i=1}^n b_i \partial_i u + cu. \tag{4.20}$$

The proof of the existence of a weak solution of (4.17) is similar to the proof for the Dirichlet Laplacian, with one exception. If L is not symmetric, we cannot use a to define an equivalent inner product on $H_0^1(\Omega)$ and appeal to the Riesz representation theorem. Instead we use a result due to Lax and Milgram which applies to non-symmetric bilinear forms.[2]

4.7. The Lax-Milgram theorem and general elliptic PDEs

We begin by stating the Lax-Milgram theorem for a bilinear form on a Hilbert space. Afterwards, we verify its hypotheses for the bilinear form associated with a general second-order uniformly elliptic PDE and use it to prove the existence of weak solutions.

THEOREM 4.20. *Let $\mathcal{H}$ be a Hilbert space with inner-product $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, and let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form on $\mathcal{H}$. Assume that there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \|u\|^2 \le a(u, u), \quad |a(u, v)| \le C_2 \|u\| \, \|v\| \qquad \text{for all } u, v \in \mathcal{H}.$$

Then for every bounded linear functional $f : \mathcal{H} \to \mathbb{R}$, there exists a unique $u \in \mathcal{H}$ such that

$$\langle f, v \rangle = a(u, v) \qquad \text{for all } v \in \mathcal{H}.$$

For the proof, see [**5**]. The verification of the hypotheses for (4.18) depends on the following energy estimates.

[2]The story behind this result — the story might be completely true or completely false — is that Lax and Milgram attended a seminar where the speaker proved existence for a symmetric PDE by use of the Riesz representation theorem, and one of them asked the other if symmetry was required; in half an hour, they convinced themselves that is wasn't, giving birth to the Lax-Milgram "lemma."

THEOREM 4.21. *Let a be the bilinear form on $H_0^1(\Omega)$ defined in (4.18), where the coefficients satisfy (4.15) and the uniform ellipticity condition (4.16) with constant θ. Then there exist constants $C_1, C_2 > 0$ and $\gamma \in \mathbb{R}$ such that for all $u, v \in H_0^1(\Omega)$*

$$C_1\|u\|_{H_0^1}^2 \leq a(u,u) + \gamma\|u\|_{L^2}^2 \tag{4.21}$$

$$|a(u,v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1}, \tag{4.22}$$

If $b = 0$, we may take $\gamma = \theta - c_0$ where $c_0 = \inf_\Omega c$, and if $b \neq 0$, we may take

$$\gamma = \frac{1}{2\theta}\sum_{i=1}^n \|b_i\|_{L^\infty}^2 + \frac{\theta}{2} - c_0.$$

PROOF. First, we have for any $u, v \in H_0^1(\Omega)$ that

$$\begin{aligned}
|a(u,v)| &\leq \sum_{i,j=1}^n \int_\Omega |a_{ij}\partial_i u \partial_j v| \, dx + \sum_{i=1}^n \int_\Omega |b_i u \partial_i v| \, dx + \int_\Omega |cuv| \, dx. \\
&\leq \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} \|\partial_i u\|_{L^2} \|\partial_j v\|_{L^2} \\
&\qquad + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i v\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\
&\leq C \left(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} + \sum_{i=1}^n \|b_i\|_{L^\infty} + \|c\|_{L^\infty} \right) \|u\|_{H_0^1} \|v\|_{H_0^1},
\end{aligned}$$

which shows (4.22).

Second, using the uniform ellipticity condition (4.16), we have

$$\begin{aligned}
\theta\|Du\|_{L^2}^2 &= \theta \int_\Omega |Du|^2 \, dx \\
&\leq \sum_{i,j=1}^n \int_\Omega a_{ij}\partial_i u \partial_j u \, dx \\
&\leq a(u,u) + \sum_{i=1}^n \int_\Omega b_i u \partial_i u \, dx - \int_\Omega cu^2 \, dx \\
&\leq a(u,u) + \sum_{i=1}^n \int_\Omega |b_i u \partial_i u| \, dx - c_0 \int_\Omega u^2 \, dx \\
&\leq a(u,u) + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i u\|_{L^2} - c_0 \|u\|_{L^2} \\
&\leq a(u,u) + \beta \|u\|_{L^2} \|Du\|_{L^2} - c_0 \|u\|_{L^2},
\end{aligned}$$

where $c(x) \geq c_0$ a.e. in Ω, and

$$\beta = \left(\sum_{i=1}^n \|b_i\|_{L^\infty}^2 \right)^{1/2}.$$

If $\beta = 0$, we get (4.21) with

$$\gamma = \theta - c_0, \qquad C_1 = \theta.$$

If $\beta > 0$, by Cauchy's inequality with ϵ, we have for any $\epsilon > 0$ that

$$\|u\|_{L^2} \|Du\|_{L^2} \leq \epsilon \|Du\|_{L^2}^2 + \frac{1}{4\epsilon} \|u\|_{L^2}^2 .$$

Hence, choosing $\epsilon = \theta/2\beta$, we get

$$\frac{\theta}{2}\|Du\|_{L^2}^2 \leq a(u,u) + \left(\frac{\beta^2}{2\theta} - c_0\right) \|u\|_{L^2} ,$$

and (4.21) follows with

$$\gamma = \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0, \qquad C_1 = \frac{\theta}{2}.$$

□

Equation (4.21) is called Gårding's inequality; this estimate of the H_0^1-norm of u in terms of $a(u,u)$, using the uniform ellipticity of L, is the crucial energy estimate. Equation (4.22) states that the bilinear form a is bounded on H_0^1. The expression for γ in this Theorem is not necessarily sharp. For example, as in the case of the Laplacian, the use of Poincaré's inequality gives smaller values of γ for bounded domains.

THEOREM 4.22. *Suppose that Ω is an open set in $\mathbb{R}^n$, and $f \in H^{-1}(\Omega)$. Let L be a differential operator (4.14) with coefficients that satisfy (4.15), and let $\gamma \in \mathbb{R}$ be a constant for which Theorem 4.21 holds. Then for every $\mu \geq \gamma$ there is a unique weak solution of the Dirichlet problem*

$$Lu + \mu f = 0, \qquad u \in H_0^1(\Omega)$$

in the sense of Definition 4.19.

PROOF. For $\mu \in \mathbb{R}$, define $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a_\mu(u,v) = a(u,v) + \mu(u,v)_{L^2} \tag{4.23}$$

where a is defined in (4.18). Then $u \in H_0^1(\Omega)$ is a weak solution of $Lu + \mu u = f$ if and only if

$$a_\mu(u,\phi) = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

From (4.22),

$$|a_\mu(u,v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1} + |\mu| \|u\|_{L^2}\|v\|_{L^2} \leq (C_2 + |\mu|) \|u\|_{H_0^1} \|v\|_{H_0^1}$$

so a_μ is bounded on $H_0^1(\Omega)$. From (4.21),

$$C_1\|u\|_{H_0^1}^2 \leq a(u,u) + \gamma\|u\|_{L^2}^2 \leq a_\mu(u,u)$$

whenever $\mu \geq \gamma$. Thus, by the Lax-Milgram theorem, for every $f \in H^{-1}(\Omega)$ there is a unique $u \in H_0^1(\Omega)$ such that $\langle f, \phi\rangle = a_\mu(u,\phi)$ for all $v \in H_0^1(\Omega)$, which proves the result. □

Although L^* is not of exactly the same form as L, since it first derivative term is not in divergence form, the same proof of the existence of weak solutions for L applies to L^* with a in (4.18) replaced by a^* in (4.19).

4.8. Compactness of the resolvent

An elliptic operator $L + \mu I$ of the type studied above is a bounded, invertible linear map from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ for sufficiently large $\mu \in \mathbb{R}$, so we may define an inverse operator $K = (L + \mu I)^{-1}$. If Ω is a bounded open set, then the Sobolev imbedding theorem implies that $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, and therefore K is a compact operator on $L^2(\Omega)$.

The operator $(L - \lambda I)^{-1}$ is called the resolvent of L, so this property is sometimes expressed by saying that L has compact resolvent. As discussed in Example 4.14, $L + \mu I$ may fail to be invertible at smaller values of μ, such that $\lambda = -\mu$ belongs to the spectrum $\sigma(L)$ of L, and the resolvent is not defined as a bounded operator on $L^2(\Omega)$ for $\lambda \in \sigma(L)$.

The compactness of the resolvent of elliptic operators on bounded open sets has several important consequences for the solvability of the elliptic PDE and the spectrum of the elliptic operator. Before describing some of these, we discuss the resolvent in more detail.

From Theorem 4.22, for $\mu \geq \gamma$ we can define

$$K : L^2(\Omega) \to L^2(\Omega), \qquad K = (L + \mu I)^{-1}\big|_{L^2(\Omega)}.$$

We define the inverse K on $L^2(\Omega)$, rather than $H^{-1}(\Omega)$, in which case its range is a subspace of $H_0^1(\Omega)$. If the domain Ω is sufficiently smooth for elliptic regularity theory to apply, then $u \in H^2(\Omega)$ if $f \in L^2(\Omega)$, and the range of K is $H^2(\Omega) \cap H_0^1(\Omega)$; for non-smooth domains, the range of K is more difficult to describe.

If we consider L as an operator acting in $L^2(\Omega)$, then the domain of L is $D = \operatorname{ran} K$, and

$$L : D \subset L^2(\Omega) \to L^2(\Omega)$$

is an unbounded linear operator with dense domain D. The operator L is closed, meaning that if $\{u_n\}$ is a sequence of functions in D such that $u_n \to u$ and $Lu_n \to f$ in $L^2(\Omega)$, then $u \in D$ and $Lu = f$. By using the resolvent, we can replace an analysis of the unbounded operator L by an analysis of the bounded operator K.

If $f \in L^2(\Omega)$, then $\langle f, v \rangle = (f, v)_{L^2}$. It follows from the definition of weak solution of $Lu + \mu u = f$ that

$$Kf = u \quad \text{if and only if} \quad a_\mu(u, v) = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega) \tag{4.24}$$

where a_μ is defined in (4.23). We also define the operator

$$K^* : L^2(\Omega) \to L^2(\Omega), \qquad K^* = (L^* + \mu I)^{-1}\big|_{L^2(\Omega)},$$

meaning that

$$K^* f = u \quad \text{if and only if} \quad a_\mu^*(u, v) = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega) \tag{4.25}$$

where $a_\mu^*(u, v) = a^*(u, v) + \mu\, (u, v)_{L^2}$ and a^* is given in (4.19).

THEOREM 4.23. *If $K \in \mathcal{B}\left(L^2(\Omega)\right)$ is defined by (4.24), then the adjoint of K is K^* defined by (4.25). If Ω is a bounded open set, then K is a compact operator.*

PROOF. If $f, g \in L^2(\Omega)$ and $Kf = u$, $K^* g = v$, then using (4.24) and (4.25), we get

$$(f, K^* g)_{L^2} = (f, v)_{L^2} = a_\mu(u, v) = a_\mu^*(v, u) = (g, u)_{L^2} = (u, g)_{L^2} = (Kf, g)_{L^2}.$$

Hence, K^* is the adjoint of K.

If $Kf = u$, then (4.21) with $\mu \geq \gamma$ and (4.24) imply that

$$C_1 \|u\|_{H_0^1}^2 \leq a_\mu(u,u) = (f,u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H_0^1}.$$

Hence $\|Kf\|_{H_0^1} \leq C\|f\|_{L^2}$ where $C = 1/C_1$. It follows that K is compact if Ω is bounded, since it maps bounded sets in $L^2(\Omega)$ into bounded sets in $H_0^1(\Omega)$, which are precompact in $L^2(\Omega)$ by the Sobolev imbedding theorem. □

4.9. The Fredholm alternative

Consider the Dirichlet problem

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.26}$$

where Ω is a smooth, bounded open set, and

$$Lu = -\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^n \partial_i \left(b_i u\right) + cu.$$

If $u = v = 0$ on $\partial\Omega$, Green's formula implies that

$$\int_\Omega (Lu) v \, dx = \int_\Omega u \left(L^* v\right) \, dx,$$

where the formal adjoint L^* of L is defined by

$$L^* v = -\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j v\right) - \sum_{i=1}^n b_i \partial_i v + cv.$$

It follows that if u is a smooth solution of (4.26) and v is a smooth solution of the homogeneous adjoint problem,

$$L^* v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then

$$\int_\Omega f v \, dx = \int_\Omega (Lu) v \, dx = \int_\Omega u L^* v \, dx = 0.$$

Thus, a necessary condition for (4.26) to be solvable is that f is orthogonal with respect to the $L^2(\Omega)$-inner product to every solution of the homogeneous adjoint problem.

For bounded domains, we will use the compactness of the resolvent to prove that this condition is necessary and sufficient for the existence of a weak solution of (4.26) where $f \in L^2(\Omega)$. Moreover, the solution is unique if and only if a solution exists for every $f \in L^2(\Omega)$.

This result is a consequence of the fact that if K is compact, then the operator $I+\sigma K$ is a Fredholm operator with index zero on $L^2(\Omega)$ for any $\sigma \in \mathbb{R}$, and therefore satisfies the Fredholm alternative (see Section 4.B.2). Thus, if $K = (L + \mu I)^{-1}$ is compact, the inverse elliptic operator $L - \lambda I$ also satisfies the Fredholm alternative.

THEOREM 4.24. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and L is a uniformly elliptic operator of the form (4.14) whose coefficients satisfy (4.15). Let L^* be the adjoint operator (4.20) and $\lambda \in \mathbb{R}$. Then one of the following two alternatives holds.*

(1) *The only weak solution of the equation $L^* v - \lambda v = 0$ is $v = 0$. For every $f \in L^2(\Omega)$ there is a unique weak solution $u \in H_0^1(\Omega)$ of the equation $Lu - \lambda u = f$. In particular, the only solution of $Lu - \lambda u = 0$ is $u = 0$.*

(2) *The equation* $L^*v - \lambda v = 0$ *has a nonzero weak solution* v. *The solution spaces of* $Lu - \lambda u = 0$ *and* $L^*v - \lambda v = 0$ *are finite-dimensional and have the same dimension. For* $f \in L^2(\Omega)$, *the equation* $Lu - \lambda u = f$ *has a weak solution* $u \in H_0^1(\Omega)$ *if and only if* $(f, v) = 0$ *for every* $v \in H_0^1(\Omega)$ *such that* $L^*v - \lambda v = 0$, *and if a solution exists it is not unique.*

PROOF. Since $K = (L + \mu I)^{-1}$ is a compact operator on $L^2(\Omega)$, the Fredholm alternative holds for the equation

$$u + \sigma K u = g \qquad u, g \in L^2(\Omega) \tag{4.27}$$

for any $\sigma \in \mathbb{R}$. Let us consider the two alternatives separately.

First, suppose that the only solution of $v + \sigma K^* v = 0$ is $v = 0$, which implies that the only solution of $L^*v + (\mu + \sigma)v = 0$ is $v = 0$. Then the Fredholm alterative for $I + \sigma K$ implies that (4.27) has a unique solution $u \in L^2(\Omega)$ for every $g \in L^2(\Omega)$. In particular, for any $g \in \operatorname{ran} K$, there exists a unique solution $u \in L^2(\Omega)$, and the equation implies that $u \in \operatorname{ran} K$. Hence, we may apply $L + \mu I$ to (4.27), and conclude that for every $f = (L + \mu I)g \in L^2(\Omega)$, there is a unique solution $u \in \operatorname{ran} K \subset H_0^1(\Omega)$ of the equation

$$Lu + (\mu + \sigma)u = f. \tag{4.28}$$

Taking $\sigma = -(\lambda + \mu)$, we get part (1) of the Fredholm alternative for L.

Second, suppose that $v + \sigma K^* v = 0$ has a finite-dimensional subspace of solutions $v \in L^2(\Omega)$. It follows that $v \in \operatorname{ran} K^*$ (clearly, $\sigma \neq 0$ in this case) and

$$L^*v + (\mu + \sigma)v = 0.$$

By the Fredholm alternative, the equation $u + \sigma K u = 0$ has a finite-dimensional subspace of solutions of the same dimension, and hence so does

$$Lu + (\mu + \sigma)u = 0.$$

Equation (4.27) is solvable for $u \in L^2(\Omega)$ given $g \in \operatorname{ran} K$ if and only if

$$(v, g)_{L^2} = 0 \qquad \text{for all } v \in L^2(\Omega) \text{ such that } v + \sigma K^* v = 0, \tag{4.29}$$

and then $u \in \operatorname{ran} K$. It follows that the condition (4.29) with $g = Kf$ is necessary and sufficient for the solvability of (4.28) given $f \in L^2(\Omega)$. Since

$$(v, g)_{L^2} = (v, Kf)_{L^2} = (K^* v, f)_{L^2} = -\frac{1}{\sigma}(v, f)_{L^2}$$

and $v + \sigma K^* v = 0$ if and only if $L^*v + (\mu + \sigma)v = 0$, we conclude that (4.28) is solvable for u if and only if $f \in L^2(\Omega)$ satisfies

$$(v, f)_{L^2} = 0 \qquad \text{for all } v \in \operatorname{ran} K \text{ such that } L^*v + (\mu + \sigma)v = 0.$$

Taking $\sigma = -(\lambda + \mu)$, we get alternative (2) for L. $\square$

Elliptic operators on a Riemannian manifold may have nonzero Fredholm index. The Atiyah-Singer index theorem (1968) relates the Fredholm index of such operators with a topological index of the manifold.

4.10. The spectrum of a self-adjoint elliptic operator

Suppose that L is a symmetric, uniformly elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + cu \tag{4.30}$$

where $a_{ij} = a_{ji}$ and $a_{ij}, c \in L^\infty(\Omega)$. The associated symmetric bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

is given by

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j u + cuv\right) dx.$$

The resolvent $K = (L + \mu I)^{-1}$ is a compact self-adjoint operator on $L^2(\Omega)$ for sufficiently large μ. Therefore its eigenvalues are real and its eigenfunctions provide an orthonormal basis of $L^2(\Omega)$. Since L has the same eigenfunctions as K, we get the corresponding result for L.

Theorem 4.25. *The operator L has an increasing sequence of real eigenvalues of finite multiplicity*

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \ldots$$

such that $\lambda_n \to \infty$. There is an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $L^2(\Omega)$ consisting of eigenfunctions functions $\phi_n \in H_0^1(\Omega)$ such that

$$L\phi_n = \lambda_n \phi_n.$$

Proof. If $K\phi = 0$ for any $\phi \in L^2(\Omega)$, then applying $L + \mu I$ to the equation we find that $\phi = 0$, so 0 is not an eigenvalue of K. If $K\phi = \kappa\phi$, for $\phi \in L^2(\Omega)$ and $\kappa \neq 0$, then $\phi \in \operatorname{ran} K$ and

$$L\phi = \left(\frac{1}{\kappa} - \mu\right)\phi,$$

so ϕ is an eigenfunction of L with eigenvalue $\lambda = 1/\kappa - \mu$. From Gårding's inequality (4.21) with $u = \phi$, and the fact that $a(\phi, \phi) = \lambda\|\phi\|_{L^2}^2$, we get

$$C_1\|\phi\|_{H_0^1}^2 \leq (\lambda + \gamma)\|\phi\|_{L^2}^2.$$

It follows that $\lambda > -\gamma$, so the eigenvalues of L are bounded from below, and at most a finite number are negative. The spectral theorem for the compact self-adjoint operator K then implies the result. □

The boundedness of the domain Ω is essential here, otherwise K need not be compact, and the spectrum of L need not consist only of eigenvalues.

Example 4.26. Suppose that $\Omega = \mathbb{R}^n$ and $L = -\Delta$. Let $K = (-\Delta + I)^{-1}$. Then, from Example 4.12, $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The range of K is $H^2(\mathbb{R}^n)$. This operator is bounded but not compact. For example, if $\phi \in C_c^\infty(\mathbb{R}^n)$ is any nonzero function and $\{a_j\}$ is a sequence in $\mathbb{R}^n$ such that $|a_j| \uparrow \infty$ as $j \to \infty$, then the sequence $\{\phi_j\}$ defined by $\phi_j(x) = \phi(x - a_j)$ is bounded in $L^2(\mathbb{R}^n)$ but $\{K\phi_j\}$ has no convergent subsequence. In this example, K has continuous spectrum $[0, 1]$ on $L^2(\mathbb{R}^n)$ and no eigenvalues. Correspondingly, $-\Delta$ has the purely continuous spectrum $[0, \infty)$.

Finally, let us briefly consider the Fredholm alternative for a self-adjoint elliptic equation from the perspective of this spectral theory. The equation

$$Lu - \lambda u = f \tag{4.31}$$

may be solved by expansion with respect to the eigenfunctions of L. Suppose that $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$ such that $L\phi_n = \lambda_n \phi_n$, where the eigenvalues λ_n are increasing and repeated according to their multiplicity. We get the following alternatives, where all series converge in $L^2(\Omega)$:

(1) If $\lambda \neq \lambda_n$ for any $n \in \mathbb{N}$, then (4.31) has the unique solution

$$u = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n$$

for every $f \in L^2(\Omega)$;

(2) If $\lambda = \lambda_M$ for for some $M \in \mathbb{N}$ and $\lambda_n = \lambda_M$ for $M \leq n \leq N$, then (4.31) has a solution $u \in H_0^1(\Omega)$ if and only if $f \in L^2(\Omega)$ satisfies

$$(f, \phi_n) = 0 \qquad \text{for } M \leq n \leq N.$$

In that case, the solutions are

$$u = \sum_{\lambda_n \neq \lambda} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n + \sum_{n=M}^{N} c_n \phi_n$$

where $\{c_M, \dots, c_N\}$ are arbitrary real constants.

4.11. Interior regularity

Roughly speaking, solutions of elliptic PDEs are as smooth as the data allows. For boundary value problems, it is convenient to consider the regularity of the solution in the interior of the domain and near the boundary separately. We begin by studying the interior regularity of solutions. We follow closely the presentation in [**5**].

To motivate the regularity theory, consider the following simple *a priori* estimate for the Laplacian. Suppose that $u \in C_c^\infty(\mathbb{R}^n)$. Then, integrating by parts twice, we get

$$\begin{aligned} \int (\Delta u)^2 \, dx &= \sum_{i,j=1}^{n} \int \left(\partial_{ii}^2 u\right) \left(\partial_{jj}^2 u\right) \, dx \\ &= -\sum_{i,j=1}^{n} \int \left(\partial_{iij}^3 u\right) \left(\partial_j u\right) \, dx \\ &= \sum_{i,j=1}^{n} \int \left(\partial_{ij}^2 u\right) \left(\partial_{ij}^2 u\right) \, dx \\ &= \int \left|D^2 u\right|^2 \, dx. \end{aligned}$$

Hence, if $-\Delta u = f$, then

$$\left\|D^2 u\right\|_{L^2} = \|f\|_{L^2}^2.$$

Thus, we can control the L^2-norm of all second derivatives of u by the L^2-norm of the Laplacian of u. This estimate suggests that we should have $u \in H^2_{\mathrm{loc}}$ if $f, u \in L^2$, as is in fact true. The above computation is, however, not justified for weak solutions that belong to H^1; as far as we know from the previous existence theory, such solutions may not even possess second-order weak derivatives.

We will consider a PDE

$$Lu = f \qquad \text{in } \Omega \tag{4.32}$$

where Ω is an open set in $\mathbb{R}^n$, $f \in L^2(\Omega)$, and L is a uniformly elliptic of the form

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right). \tag{4.33}$$

It is straightforward to extend the proof of the regularity theorem to uniformly elliptic operators that contain lower-order terms [**5**].

A function $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33) if

$$a(u, v) = (f, v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{4.34}$$

where the bilinear form a is given by

$$a(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_i u \partial_j v \, dx. \tag{4.35}$$

We do not impose any boundary condition on u, for example by requiring that $u \in H_0^1(\Omega)$, so the interior regularity theorem applies to any weak solution of (4.32).

Before stating the theorem, we illustrate the idea of the proof with a further *a priori* estimate. To obtain a local estimate for $D^2 u$ on a subdomain $\Omega' \Subset \Omega$, we introduce a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω'. We take as a test function

$$v = -\partial_k \eta^2 \partial_k u. \tag{4.36}$$

Note that v is given by a positive-definite, symmetric operator acting on u of a similar form to L, which leads to the positivity of the resulting estimate for $D\partial_k u$.

Multiplying (4.32) by v and integrating over Ω, we get $(Lu, v) = (f, v)$. Two integrations by parts imply that

$$\begin{aligned}
(Lu, v) &= \sum_{i,j=1}^{n} \int_\Omega \partial_j \left(a_{ij}\partial_i u\right) \left(\partial_k \eta^2 \partial_k u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int_\Omega \partial_k \left(a_{ij}\partial_i u\right) \left(\partial_j \eta^2 \partial_k u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int_\Omega \eta^2 a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_j \partial_k u\right) \, dx + F
\end{aligned}$$

where

$$\begin{aligned}
F = \sum_{i,j=1}^{n} \int_\Omega \Big\{ &\eta^2 \left(\partial_k a_{ij}\right) \left(\partial_i u\right) \left(\partial_j \partial_k u\right) \\
&+ 2\eta \partial_j \eta \Big[a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_k u\right) + \left(\partial_k a_{ij}\right) \left(\partial_i u\right) \left(\partial_k u\right) \Big] \Big\} \, dx.
\end{aligned}$$

The term F is linear in the second derivatives of u. We use the uniform ellipticity of L to get

$$\theta \int_{\Omega'} |D\partial_k u|^2 \, dx \leq \sum_{i,j=1}^{n} \int_{\Omega} \eta^2 a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_j \partial_k u\right) \, dx = (f, v) - F,$$

and a Cauchy inequality with ϵ to absorb the linear terms in second derivatives on the right-hand side into the quadratic terms on the left-hand side. This results in an estimate of the form

$$\|D\partial_k u\|^2_{L^2(\Omega')} \leq C \left(f^2 + \|u\|^2_{H^1(\Omega)} \right).$$

The proof of regularity is entirely analogous, with the derivatives in the test function (4.36) replaced by difference quotients (see Section 4.C). We obtain an $L^2(\Omega')$-bound for the difference quotients $D\partial_k^h u$ that is uniform in h, which implies that $u \in H^2(\Omega')$.

THEOREM 4.27. *Suppose that Ω is an open set in $\mathbb{R}^n$. Assume that $a_{ij} \in C^1(\Omega)$ and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Furthermore,*

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right) \tag{4.37}$$

where the constant C depends only on n, Ω', Ω and a_{ij}.

PROOF. Choose a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on Ω'. We use the compactly supported test function

$$v = -D_k^{-h} \left(\eta^2 D_k^h u \right) \in H_0^1(\Omega)$$

in the definition (4.34)–(4.35) for weak solutions. (As in (4.36), v is given by a positive self-adjoint operator acting on u.) This implies that

$$- \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \left(\partial_i u\right) D_k^{-h} \partial_j \left(\eta^2 D_k^h u \right) \, dx = - \int_{\Omega} f D_k^{-h} \left(\eta^2 D_k^h u \right) \, dx. \tag{4.38}$$

Performing a discrete integration by parts and using the product rule, we may write the left-hand side of (4.38) as

$$\begin{aligned} - \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \left(\partial_i u\right) D_k^{-h} \partial_j \left(\eta^2 D_k^h u \right) \, dx &= \sum_{i,j=1}^{n} \int_{\Omega} D_k^h \left(a_{ij} \partial_i u \right) \partial_j \left(\eta^2 D_k^h u \right) \, dx \\ &= \sum_{i,j=1}^{n} \int_{\Omega} \eta^2 a_{ij} \left(D_k^h \partial_i u \right) \left(D_k^h \partial_j u \right) \, dx + F, \end{aligned} \tag{4.39}$$

where, with $a_{ij}^h(x) = a_{ij}(x + h e_k)$,

$$\begin{aligned} F = \sum_{i,j=1}^{n} \int_{\Omega} \Big\{ &\eta^2 \left(D_k^h a_{ij} \right) \left(\partial_i u \right) \left(D_k^h \partial_j u \right) \\ &+ 2\eta \partial_j \eta \Big[a_{ij}^h \left(D_k^h \partial_i u \right) \left(D_k^h u \right) + \left(D_k^h a_{ij} \right) \left(\partial_i u \right) \left(D_k^h u \right) \Big] \Big\} \, dx. \end{aligned} \tag{4.40}$$

Using the uniform ellipticity of L in (4.16), we estimate

$$\theta \int_\Omega \eta^2 \left|D_k^h Du\right|^2\, dx \leq \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij} \left(D_k^h \partial_i u\right)\left(D_k^h \partial_j u\right)\, dx.$$

Using (4.38)–(4.39) and this inequality, we find that

$$\theta \int_\Omega \eta^2 \left|D_k^h Du\right|^2\, dx \leq -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\, dx - F. \tag{4.41}$$

By the Cauchy-Schwartz inequality,

$$\left|\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\, dx\right| \leq \|f\|_{L^2(\Omega)} \left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)}.$$

Since $\operatorname{spt}\eta \Subset \Omega$, Proposition 4.52 implies that for sufficiently small h,

$$\begin{aligned}
\left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} &\leq \left\|\partial_k\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} \\
&\leq \left\|\eta^2 \partial_k D_k^h u\right\|_{L^2(\Omega)} + \left\|2\eta\left(\partial_k \eta\right) D_k^h u\right\|_{L^2(\Omega)} \\
&\leq \left\|\eta \partial_k D_k^h u\right\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)}.
\end{aligned}$$

A similar estimate of F in (4.40) gives

$$|F| \leq C\left(\|Du\|_{L^2(\Omega)} \left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2\right).$$

Using these results in (4.41), we find that

$$\begin{aligned}
\theta\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 \leq & C\Big(\|f\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} \\
& + \|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2\Big).
\end{aligned} \tag{4.42}$$

By Cauchy's inequality with ϵ, we have

$$\begin{aligned}
\|f\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} &\leq \epsilon\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon}\|f\|_{L^2(\Omega)}^2, \\
\|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} &\leq \epsilon\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon}\|Du\|_{L^2(\Omega)}^2.
\end{aligned}$$

Hence, choosing ϵ so that $4C\epsilon = \theta$, and using the result in (4.42) we get that

$$\frac{\theta}{4}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 \leq C\Big(\|f\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\Big).$$

Thus, since $\eta = 1$ on Ω',

$$\left\|D_k^h Du\right\|_{L^2(\Omega')}^2 \leq C\Big(\|f\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\Big) \tag{4.43}$$

where the constant C depends on Ω, Ω', a_{ij}, but is independent of h, u, f.

We can further estimate $\|Du\|$ in terms of $\|u\|$ by taking $v = u$ in (4.34)–(4.35) and using the uniform ellipticity of L to get

$$\begin{aligned}\theta \int_\Omega |Du|^2\,dx &\le \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j u \\ &\le \int_\Omega f u\,dx \\ &\le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Using this result in (4.43), we get that

$$\left\|D_k^h Du\right\|^2_{L^2(\Omega')} \le C\Big(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big).$$

Theorem 4.53 theorem now implies that the weak second derivatives of u exist and belong to $L^2(\Omega)$. Furthermore, the H^2-norm of u satisfies (4.37). □

If $u \in H^2_{\mathrm{loc}}(\Omega)$ and $f \in L^2(\Omega)$, then the equation $Lu = f$ relating the weak derivatives of u and f holds pointwise a.e.; such solutions are often called strong solutions, to distinguish them from weak solutions which may not possess weak second order derivatives and classical solutions which possess continuous second order derivatives.

The repeated application of these estimates leads to higher interior regularity.

Theorem 4.28. *Suppose that $a_{ij} \in C^{k+1}(\Omega)$ and $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^{k+2}(\Omega')$ for every $\Omega' \Subset \Omega$. Furthermore,*

$$\|u\|_{H^{k+2}(\Omega')} \le C\left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, k, Ω', Ω and a_{ij}.

See [**5**] for a detailed proof. Note that if the above conditions hold with $k > n/2$, then $f \in C(\Omega)$ and $u \in C^2(\Omega)$, so u is a classical solution of the PDE $Lu = f$. Furthermore, if f and a_{ij} are smooth then so is the solution.

Corollary 4.29. *If $a_{ij}, f \in C^\infty(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in C^\infty(\Omega)$*

Proof. If $\Omega' \Subset \Omega$, then $f \in H^k(\Omega')$ for every $k \in \mathbb{N}$, so by Theorem (4.28) $u \in H^{k+2}_{\mathrm{loc}}(\Omega')$ for every $k \in \mathbb{N}$, and by the Sobolev imbedding theorem $u \in C^\infty(\Omega')$. Since this holds for every open set $\Omega' \Subset \Omega$, we have $u \in C^\infty(\Omega)$. □

4.12. Boundary regularity

To study the regularity of solutions near the boundary, we localize the problem to a neighborhood of a boundary point by use of a partition of unity: We decompose the solution into a sum of functions that are compactly supported in the sets of a suitable open cover of the domain and estimate each function in the sum separately.

Assuming, as in Section 1.10, that the boundary is at least C^1, we may 'flatten' the boundary in a neighborhood U by a diffeomorphism $\varphi : U \to V$ that maps $U \cap \Omega$ to an upper half space $V = B_1(0) \cap \{y_n > 0\}$. If $\varphi^{-1} = \psi$ and $x = \psi(y)$, then by a change of variables (*c.f.* Theorem 1.38 and Proposition 3.20) the weak formulation (4.32)–(4.33) on U becomes

$$\sum_{i,j=1}^{n} \int_V \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial y_i} \frac{\partial \tilde{v}}{\partial y_j}\, dy = \int_V \tilde{f} \tilde{v}\, dy \qquad \text{for all functions } \tilde{v} \in H_0^1(V),$$

where $\tilde{u} \in H^1(V)$. Here, $\tilde{u} = u \circ \psi$, $\tilde{v} = v \circ \psi$, and

$$\tilde{a}_{ij} = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \left(\frac{\partial \varphi_i}{\partial x_p} \circ \psi \right) \left(\frac{\partial \varphi_j}{\partial x_q} \circ \psi \right), \qquad \tilde{f} = |\det D\psi|\, f \circ \psi.$$

The matrix $\tilde{a}_{ij}$ satisfies the uniform ellipticity condition if a_{pq} does. To see this, we define $\zeta = (D\varphi^t)\,\xi$, or

$$\zeta_p = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_p} \xi_i.$$

Then, since $D\varphi$ and $D\psi = D\varphi^{-1}$ are invertible and bounded away from zero, we have for some constant $C > 0$ that

$$\sum_{i,j}^{n} \tilde{a}_{ij} \xi_i \xi_j = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \zeta_p \zeta_q \geq |\det D\psi|\ \theta |\zeta|^2 \geq C\theta |\xi|^2.$$

Thus, we obtain a problem of the same form as before after the change of variables. Note that we must require that the boundary is C^2 to ensure that $\tilde{a}_{ij}$ is C^1.

It is important to recognize that in changing variables for weak solutions, we need to verify the change of variables for the weak formulation directly and not for the original PDE. A transformation that is valid for smooth solutions of a PDE is not always valid for weak solutions, which may lack sufficient smoothness to justify the transformation.

We now state a boundary regularity theorem. Unlike the interior regularity theorem, we impose a boundary condition $u \in H_0^1(\Omega)$ on the solution, and we require that the boundary of the domain is smooth. A solution of an elliptic PDE with smooth coefficients and smooth right-hand side is smooth in the interior of its domain of definition, whatever its behavior near the boundary; but we cannot expect to obtain smoothness up to the boundary without imposing a smooth boundary condition on the solution and requiring that the boundary is smooth.

THEOREM 4.30. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^2-boundary. Assume that $a_{ij} \in C^1(\overline{\Omega})$ and $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^2(\Omega)$, and*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

where the constant C depends only on n, Ω and a_{ij}.

PROOF. By use of a partition of unity and a flattening of the boundary, it is sufficient to prove the result for an upper half space $\Omega = \{(x_1, \dots, x_n) : x_n > 0\}$ space and functions $u, f : \Omega \to \mathbb{R}$ that are compactly supported in $B_1(0) \cap \overline{\Omega}$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_1(0)$. We will estimate the tangential and normal difference quotients of Du separately.

First consider a test function that depends on tangential differences,

$$v = -D_k^{-h} \eta^2 D_k^h u \qquad \text{for } k = 1, 2, \dots, n-1.$$

Since the trace of u is zero on $\partial\Omega$, the trace of v on $\partial\Omega$ is zero and, by Theorem 3.42, $v \in H_0^1(\Omega)$. Thus we may use v in the definition of weak solution to get (4.38). Exactly the same argument as the one in the proof of Theorem 4.27 gives (4.43). It follows from Theorem 4.53 that the weak derivatives $\partial_k \partial_i u$ exist and satisfy

$$\|\partial_k Du\|_{L^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right) \qquad \text{for } k = 1, 2, \dots, n-1. \tag{4.44}$$

The only derivative that remains is the second-order normal derivative $\partial_n^2 u$, which we can estimate from the equation. Using (4.32)–(4.33), we have for $\phi \in C_c^\infty(\Omega)$ that

$$\int_\Omega a_{nn}\left(\partial_n u\right)\left(\partial_n \phi\right)\, dx = -\sum{}' \int_\Omega a_{ij}\left(\partial_i u\right)\left(\partial_j \phi\right)\, dx + \int_\Omega f\phi\, dx$$

where $\sum'$ denotes the sum over $1 \le i, j \le n$ with the term $i = j = n$ omitted. Since $a_{ij} \in C^1(\Omega)$ and $\partial_i u$ is weakly differentiable with respect to x_j unless $i = j = n$ we get, using Proposition 3.20, that

$$\int_\Omega a_{nn}\left(\partial_n u\right)\left(\partial_n \phi\right)\, dx = \sum{}' \int_\Omega \left\{\partial_j\left[a_{ij}\left(\partial_i u\right)\right] + f\right\}\phi\, dx \qquad \text{for every } \phi \in C_c^\infty(\Omega).$$

It follows that $a_{nn}\left(\partial_n u\right)$ is weakly differentiable with respect to x_n, and

$$\partial_n\left[a_{nn}\left(\partial_n u\right)\right] = -\left\{\sum{}' \partial_j\left[a_{ij}\left(\partial_i u\right)\right] + f\right\} \in L^2(\Omega).$$

From the uniform ellipticity condition (4.16) with $\xi = e_n$, we have $a_{nn} \ge \theta$. Hence, by Proposition 3.20,

$$\partial_n u = \frac{1}{a_{nn}}\, a_{nn}\partial_n u$$

is weakly differentiable with respect to x_n with derivative

$$\partial_{nn}^2 u = \frac{1}{a_{nn}}\partial_n\left[a_{nn}\partial_n u\right] + \partial_n\left(\frac{1}{a_{nn}}\right) a_{nn}\partial_n u \in L^2(\Omega).$$

Furthermore, using (4.44) we get an estimate of the same form for $\|\partial_{nn}^2 u\|_{L^2(\Omega)}^2$, so that

$$\left\|D^2 u\right\|_{L^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)$$

□

The repeated application of these estimates leads to higher-order regularity.

THEOREM 4.31. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^{k+2}-boundary. Assume that $a_{ij} \in C^{k+1}(\overline{\Omega})$ and $f \in H^k(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^{k+2}(\Omega)$ and*

$$\|u\|_{H^{k+2}(\Omega)} \le C\left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, k, Ω, and a_{ij}.

Sobolev imbedding then yields the following result.

COROLLARY 4.32. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^∞ boundary. If $a_{ij}, f \in C^\infty(\overline{\Omega})$ and $u \in H_0^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in C^\infty(\overline{\Omega})$*

4.13. Some further perspectives

The above results give an existence and L^2-regularity theory for second-order, uniformly elliptic PDEs in divergence form. This theory is based on the simple *a priori* energy estimate for $\|Du\|_{L^2}$ that we obtain by multiplying the equation $Lu = f$ by u, or some derivative of u, and integrating the result by parts.

This theory is a fundamental one, but there is a bewildering variety of approaches to the existence and regularity of solutions of elliptic PDEs. In an attempt to put the above analysis in a broader context, we briefly list some of these approaches and other important results, without any claim to completeness. Many of these topics are discussed further in the references [**5, 10, 12**].

L^p-theory: If $1 < p < \infty$, there is a similar regularity result that solutions of $Lu = f$ satisfy $u \in W^{2,p}$ if $f \in L^p$. The derivation is not as simple when $p \neq 2$, however, and requires the use of more sophisticated tools from real analysis (such as Calderón-Zygmund operators in harmonic analysis).

Schauder theory: The Schauder theory provides Hölder-estimates similar to those derived in Section 2.7.2 for Laplace's equation, and a corresponding existence theory of solutions $u \in C^{2,\alpha}$ of $Lu = f$ if $f \in C^{0,\alpha}$ and L has Hölder continuous coefficients. General linear elliptic PDEs are treated by regarding them as perturbations of constant coefficient PDEs, an approach that works because there is no 'loss of derivatives' in the estimates of the solution. The Hölder estimates were originally obtained by the use of potential theory, but other ways to obtain them are now known; for example, by the use of Campanato spaces, which provide Hölder norms in terms of suitable integrals that are easier to estimate directly.

Perron's method: Perron (1923) showed that solutions of the Dirichlet problem for Laplace's equation can be obtained as the infimum of superharmonic functions or the supremum of subharmonic functions, together with the use of barrier functions to prove that, under suitable assumptions on the boundary, the solution attains the prescribed boundary values. This method is based on maximum principle estimates.

Boundary integral methods: By the use of Green's functions, one can often reduce a linear elliptic BVP to an integral equation on the boundary, and then use the theory of integral equations to study the existence and regularity of solutions. These methods also provide efficient numerical schemes because of the lower dimensionality of the boundary.

Pseudo-differential operators: The Fourier transform provides an effective method for solving linear PDEs with constant coefficients. The theory of pseudo-differential and Fourier-integral operators is a powerful extension of this method that applies to general linear PDEs with variable coefficients, and elliptic PDEs in particular. It is, however, less well-suited to the analysis of nonlinear PDEs.

Variational methods: Many elliptic PDEs — especially those in divergence form — arise as Euler-Lagrange equations for variational principles. Existence of weak solutions can often be shown by use of the direct method of the calculus of variations, after which one studies the regularity of a minimizer (or, in some cases, a critical point).

Di Giorgi-Nash-Moser: The work of Di Giorgi (1957), Nash (1958), and Moser (1960) showed that weak solutions of a second order elliptic PDE in divergence form with bounded (L^∞) coefficients are Hölder continuous ($C^{0,\alpha}$). This was the key step in developing a regularity theory for minimizers of nonlinear variational principles with elliptic Euler-Lagrange equations. Moser also obtained a Harnack inequality for weak solutions.

Fully nonlinear equations: Krylov and Safonov (1979) obtained a Harnack inequality for second order elliptic equations in nondivergence form. This allowed the development of a regularity theory for fully nonlinear elliptic equations (*e.g.* second-order equations for u that depend nonlinearly on D^2u). Crandall and Lions (1983) introduced the notion of viscosity solutions which — despite the name — uses the maximum principle and is based on a comparison with appropriate sub and super solutions This theory applies to fully nonlinear elliptic PDEs, although it is mainly restricted to scalar equations.

Degree theory: Topological methods based on the Leray-Schauder degree of a mapping on a Banach space can be used to prove existence of solutions of various nonlinear elliptic problems (see *e.g.* L. Nirenberg, *Topics in Nonlinear Functional Analysis*). These methods can provide global existence results for large solutions, but often do not give much detailed analytical information about the solutions.

Heat flow methods: Parabolic PDEs, such as $u_t + Lu = f$, are closely connected with the associated elliptic PDEs for stationary solutions, such as $Lu = f$. One may use this connection to obtain solutions of an elliptic PDE as the limit as $t \to \infty$ of solutions of the associated parabolic PDE. For example, Hamilton (1981) introduced the Ricci flow on a manifold, in which the metric approaches a Ricci-flat metric as $t \to \infty$, as a means to understand the topological classification of smooth manifolds, and Perelman (2003) used this approach to prove the Poincaré conjecture (that every simply connected, three-dimensional, compact manifold without boundary is homeomorphic to a three-dimensional sphere) and, more generally, the geometrization conjecture of Thurston.