4.A. Heat flow

As a simple application that leads to second order PDEs, we consider the problem of finding the temperature distribution inside a body. Similar equations describe the diffusion of a solute. Steady temperature distributions satisfy an elliptic PDE, such as Laplace's equation, while unsteady distributions satisfy a parabolic PDE, such as the heat equation.

4.A.1. Steady heat flow. Suppose that the body occupies an open set Ω in $\mathbb{R}^n$. Let $u : \Omega \to \mathbb{R}$ denote the temperature, $g : \Omega \to \mathbb{R}$ the rate per unit volume at which heat sources create energy inside the body, and $\vec{q} : \Omega \to \mathbb{R}^n$ the heat flux. That is, the rate per unit area at which heat energy diffuses across a surface with normal $\vec{\nu}$ is equal to $\vec{q} \cdot \vec{\nu}$.

If the temperature distribution is steady, then conservation of energy implies that for any smooth open set $\Omega' \Subset \Omega$ the heat flux out of Ω' is equal to the rate at which heat energy is generated inside Ω'; that is,

$$\int_{\partial\Omega'} \vec{q} \cdot \vec{\nu}\, dS = \int_{\Omega'} g\, dV.$$

Here, we use dS and dV to denote integration with respect to surface area and volume, respectively.

We assume that $\vec{q}$ and g are smooth. Then, by the divergence theorem,

$$\int_{\Omega'} \operatorname{div} \vec{q}\, dV = \int_{\Omega'} g\, dV.$$

Since this equality holds for all subdomains Ω' of Ω, it follows that

$$\operatorname{div} \vec{q} = g \qquad \text{in } \Omega. \tag{4.45}$$

Equation (4.45) expresses the fundamental physical principle of conservation of energy, but this principle alone is not enough to determine the temperature distribution inside the body. We must supplement it with a constitutive relation that describes how the heat flux is related to the temperature distribution.

Fourier's law states that the heat flux at some point of the body depends linearly on the temperature gradient at the same point and is in a direction of decreasing temperature. This law is an excellent and well-confirmed approximation in a wide variety of circumstances. Thus,

$$\vec{q} = -A\nabla u \tag{4.46}$$

for a suitable conductivity tensor $A : \Omega \to L(\mathbb{R}^n, \mathbb{R}^n)$, which is required to be symmetric and positive definite. Explicitly, if $\vec{x} \in \Omega$, then $A(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map that takes the negative temperature gradient at $\vec{x}$ to the heat flux at $\vec{x}$. In a uniform, isotropic medium $A = \kappa I$ where the constant $\kappa > 0$ is the thermal conductivity. In an anisotropic medium, such as a crystal or a composite medium, A is not proportional to the identity I and the heat flux need not be in the same direction as the temperature gradient.

Using (4.46) in (4.45), we find that the temperature u satisfies

$$-\operatorname{div}(A\nabla u) = g.$$

If we denote the matrix of A with respect to the standard basis in $\mathbb{R}^n$ by (a_{ij}), then the component form of this equation is

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) = g.$$

This equation is in divergence or conservation form. For smooth functions $a_{ij} : \Omega \to \mathbb{R}$, we can write it in nondivergence form as

$$-\sum_{i,j=1}^{n} a_{ij}\partial_{ij}u - \sum_{j=1}^{n} b_j\partial_j u = g, \qquad b_j = \sum_{i}^{n} \partial_i a_{ij}.$$

These forms need not be equivalent if the coefficients a_{ij} are not smooth. For example, in a composite medium made up of different materials, a_{ij} may be discontinuous across boundaries that separate the materials. Such problems can be rewritten as smooth PDEs within domains occupied by a given material, together with appropriate jump conditions across the boundaries. The weak formulation incorporates both the PDEs and the jump conditions.

Next, suppose that the body is occupied by a fluid which, in addition to conducting heat, is in motion with velocity $\vec{v} : \Omega \to \mathbb{R}^n$. Let $e : \Omega \to \mathbb{R}$ denote the internal thermal energy per unit volume of the body, which we assume is a function of the location $\vec{x} \in \Omega$ of a point in the body. Then, in addition to the diffusive flux $\vec{q}$, there is a convective thermal energy flux equal to $e\vec{v}$, and conservation of energy gives

$$\int_{\partial\Omega'} \left(\vec{q} + e\vec{v}\right) \cdot \vec{\nu}\, dS = \int_{\Omega'} g\, dV.$$

Using the divergence theorem as before, we find that

$$\operatorname{div}\left(\vec{q} + e\vec{v}\right) = g,$$

If we assume that $e = c_p u$ is proportional to the temperature, where c_p is the heat capacity per unit volume of the material in the body, and Fourier's law, we get the PDE

$$-\operatorname{div}\left(A\nabla u\right) + \operatorname{div}(\vec{b}u) = g.$$

where $\vec{b} = c_p\vec{v}$.

Suppose that $g = f - cu$ where $f : \Omega \to \mathbb{R}$ is a given energy source and cu represents a linear growth or decay term with coefficient $c : \Omega \to \mathbb{R}$. For example, lateral heat loss at a rate proportional the temperature would give decay $(c > 0)$, while the effects of an exothermic temperature-dependent chemical reaction might be approximated by a linear growth term $(c < 0)$. We then get the linear PDE

$$-\operatorname{div}\left(A\nabla u\right) + \operatorname{div}(\vec{b}u) + cu = f,$$

or in component form with $\vec{b} = (b_1, \ldots, b_n)$

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu = f.$$

This PDE describes a thermal equilibrium due to the combined effects of diffusion with diffusion matrix a_{ij}, advection with normalized velocity b_i, growth or decay with coefficient c, and external sources with density f.

In the simplest case where, after nondimensionalization, $A = I$, $\vec{b} = 0$, $c = 0$, and $f = 0$, we get Laplace's equation $\Delta u = 0$.

4.A.2. Unsteady heat flow. Consider a time-dependent heat flow in a region Ω with temperature $u(\vec{x}, t)$, energy density per unit volume $e(\vec{x}, t)$, heat flux $\vec{q}(\vec{x}, t)$, advection velocity $\vec{v}(\vec{x}, t)$, and heat source density $g(\vec{x}, t)$. Conservation of energy implies that for any subregion $\Omega' \Subset \Omega$

$$\frac{d}{dt}\int_{\Omega'} e\, dV = -\int_{\partial\Omega'} (\vec{q} + e\vec{v}) \cdot \vec{\nu}\, dS + \int_{\Omega'} g\, dV.$$

Since

$$\frac{d}{dt}\int_{\Omega'} e\, dV = \int_{\Omega'} e_t\, dV,$$

the use of the divergence theorem and the same constitutive assumptions as in the steady case lead to the parabolic PDE

$$(c_p u)_t - \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu = f.$$

In the simplest case where, after nondimensionalization, $c_p = 1$, $A = I$, $\vec{b} = 0$, $c = 0$, and $f = 0$, we get the heat equation $u_t = \Delta u$.

4.B. Operators on Hilbert spaces

Suppose that $\mathcal{H}$ is a Hilbert space with inner product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$. We denote the space of bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. This is a Banach space with respect to the operator norm, defined by

$$\|T\| = \sup_{\substack{x \in \mathcal{H} \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

The adjoint $T^* \in \mathcal{B}(\mathcal{H})$ of $T \in \mathcal{B}(\mathcal{H})$ is the linear operator such that

$$(Tx, y) = (x, T^*y) \qquad \text{for all } x, y \in \mathcal{H}.$$

An operator T is self-adjoint if $T = T^*$. The kernel and range of $T \in \mathcal{B}(\mathcal{H})$ are the subspaces

$$\ker T = \{x \in \mathcal{H} : Tx = 0\}, \quad \operatorname{ran} T = \{y \in \mathcal{H} : y = Tx \text{ for some } x \in \mathcal{H}\}.$$

We denote by $\ell^2(\mathbb{N})$, or ℓ^2 for short, the Hilbert space of square summable real sequences

$$\ell^2(\mathbb{N}) = \left\{(x_1, x_2, x_3, \dots, x_n, \dots) : x_n \in \mathbb{R} \text{ and } \textstyle\sum_{n\in\mathbb{N}} x_n^2 < \infty\right\}$$

with the standard inner product. Any infinite-dimensional, separable Hilbert space is isomorphic to ℓ^2.

4.B.1. Compact operators.

Definition 4.33. A linear operator $T \in \mathcal{B}(\mathcal{H})$ is compact if it maps bounded sets to precompact sets.

That is, T is compact if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in $\mathcal{H}$.

Example 4.34. A bounded linear map with finite-dimensional range is compact. In particular, every linear operator on a finite-dimensional Hilbert space is compact.

EXAMPLE 4.35. The identity map $I \in \mathcal{B}(\mathcal{H})$ given by $I : x \mapsto x$ is compact if and only if $\mathcal{H}$ is finite-dimensional.

EXAMPLE 4.36. The map $K \in \mathcal{B}\left(\ell^2\right)$ given by

$$K : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots, \frac{1}{n}x_n, \dots\right)$$

is compact (and self-adjoint).

We have the following spectral theorem for compact self-adjoint operators.

THEOREM 4.37. *Let $T : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator. Then T has a finite or countably infinite number of distinct nonzero, real eigenvalues. If there are infinitely many eigenvalues $\{\lambda_n \in \mathbb{R} : n \in \mathbb{N}\}$ then $\lambda_n \to 0$ as $n \to \infty$. The eigenspace associated with each nonzero eigenvalue is finite-dimensional, and eigenvectors associated with distinct eigenvalues are orthogonal. Furthermore, $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T, including those (if any) with eigenvalue zero.*

4.B.2. Fredholm operators. We summarize the definition and properties of Fredholm operators and give some examples. For proofs, see

DEFINITION 4.38. A linear operator $T \in \mathcal{B}(\mathcal{H})$ is Fredholm if: (a) $\ker T$ has finite dimension; (b) $\operatorname{ran} T$ is closed and has finite codimension.

Condition (b) and the projection theorem for Hilbert spaces imply that $\mathcal{H} = \operatorname{ran} T \oplus (\operatorname{ran} T)^\perp$ where the dimension of $\operatorname{ran} T^\perp$ is finite, and

$$\operatorname{codim} \operatorname{ran} T = \dim(\operatorname{ran} T)^\perp.$$

DEFINITION 4.39. If $T \in \mathcal{B}(\mathcal{H})$ is Fredholm, then the index of T is the integer

$$\operatorname{ind} T = \dim \ker T - \operatorname{codim} \operatorname{ran} T.$$

EXAMPLE 4.40. Every linear operator $T : \mathcal{H} \to \mathcal{H}$ on a finite-dimensional Hilbert space $\mathcal{H}$ is Fredholm and has index zero. The range is closed since every finite-dimensional linear space is closed, and the dimension formula

$$\dim \ker T + \dim \operatorname{ran} T = \dim \mathcal{H}$$

implies that the index is zero.

EXAMPLE 4.41. The identity map I on a Hilbert space of any dimension is Fredholm, with $\dim \ker P = \operatorname{codim} \operatorname{ran} P = 0$ and $\operatorname{ind} I = 0$.

EXAMPLE 4.42. The self-adjoint projection P on ℓ^2 given by

$$P : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_2, x_3, \dots, x_n, \dots)$$

is Fredholm, with $\dim \ker P = \operatorname{codim} \operatorname{ran} P = 1$ and $\operatorname{ind} P = 0$. The complementary projection

$$Q : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_1, 0, 0, \dots, 0, \dots)$$

is not Fredholm, although the range of Q is closed, since $\dim \ker Q$ and $\operatorname{codim} \operatorname{ran} Q$ are infinite.

EXAMPLE 4.43. The left and right shift maps on ℓ^2, given by

$$S : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_2, x_3, x_4, \dots, x_{n+1}, \dots),$$
$$T : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_1, x_2, \dots, x_{n-1}, \dots),$$

are Fredholm. Note that $S^* = T$. We have $\dim\ker S = 1$, $\operatorname{codim}\operatorname{ran} S = 0$, and $\dim\ker T = 0$, $\operatorname{codim}\operatorname{ran} T = 1$, so

$$\operatorname{ind} S = 1, \qquad \operatorname{ind} T = -1.$$

If $n \in \mathbb{N}$, then $\operatorname{ind} S^n = n$ and $\operatorname{ind} T^n = -n$, so the index of a Fredholm operator on an infinite-dimensional space can take all integer values. Unlike the finite-dimensional case, where a linear operator $A : \mathcal{H} \to \mathcal{H}$ is one-to-one if and only if it is onto, S fails to be one-to-one although it is onto, and T fails to be onto although it is one-to-one.

The above example also illustrates the following theorem.

THEOREM 4.44. *If $T \in \mathcal{B}(\mathcal{H})$ is Fredholm, then T^* is Fredholm with*

$$\dim\ker T^* = \operatorname{codim}\operatorname{ran} T, \quad \operatorname{codim}\operatorname{ran} T^* = \dim\ker T, \quad \operatorname{ind} T^* = -\operatorname{ind} T.$$

EXAMPLE 4.45. The compact map K in Example 4.36 is not Fredholm since the range of K,

$$\operatorname{ran} K = \left\{ (y_1, y_2, y_3, \dots, y_n, \dots) \in \ell^2 : \sum_{n\in\mathbb{N}} n^2 y_n^2 < \infty \right\},$$

is not closed. The range is dense in ℓ^2 but, for example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2 \setminus \operatorname{ran} K.$$

We denote the set of Fredholm operators by $\mathcal{F}$. Then, according to the next theorem, $\mathcal{F}$ is an open set in $\mathcal{B}(\mathcal{H})$, and

$$\mathcal{F} = \bigcup_{n\in\mathbb{Z}} \mathcal{F}_n$$

is the union of connected components $\mathcal{F}_n$ consisting of the Fredholm operators with index n. Moreover, if $T \in \mathcal{F}_n$, then $T + K \in \mathcal{F}_n$ for any compact operator K.

THEOREM 4.46. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is Fredholm and $K \in \mathcal{B}(\mathcal{H})$ is compact.*

(1) *There exists $\epsilon > 0$ such that $T + H$ is Fredholm for any $H \in \mathcal{B}(\mathcal{H})$ with $\|H\| < \epsilon$. Moreover,* $\operatorname{ind}(T + H) = \operatorname{ind} T$.
(2) *$T + K$ is Fredholm and* $\operatorname{ind}(T + K) = \operatorname{ind} T$.

Solvability conditions for Fredholm operators are a consequences of following theorem.

THEOREM 4.47. *If $T \in \mathcal{B}(\mathcal{H})$, then $\mathcal{H} = \overline{\operatorname{ran} T} \oplus \ker T^*$ and $\overline{\operatorname{ran} T} = (\ker T)^\perp$.*

Thus, if $T \in \mathcal{B}(\mathcal{H})$ has closed range, then $Tx = y$ has a solution $x \in \mathcal{H}$ if and only if $y \perp z$ for every $z \in \mathcal{H}$ such that $T^*z = 0$. For a Fredholm operator, this is finitely many linearly independent solvability conditions.

EXAMPLE 4.48. If S, T are the shift maps defined in Example 4.43, then $\ker S^* = \ker T = 0$ and the equation $Sx = y$ is solvable for every $y \in \ell^2$. Solutions are not, however, unique since $\ker S \neq 0$. The equation $Tx = y$ is solvable only if $y \perp \ker S$. If it exists, the solution is unique.

EXAMPLE 4.49. The compact map K in Example 4.36 is self adjoint, $K = K^*$, and $\ker K = 0$. Thus, every element $y \in \ell^2$ is orthogonal to $\ker K^*$, but this condition is not sufficient to imply the solvability of $Kx = y$ because the range of K os not closed. For example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2 \setminus \operatorname{ran} K.$$

For Fredholm operators with index zero, we get the following Fredholm alternative, which states that the corresponding linear equation has solvability properties which are similar to those of a finite-dimensional linear system.

THEOREM 4.50. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator and* $\operatorname{ind} T = 0$. *Then one of the following two alternatives holds:*

(1) $\ker T^* = 0$; $\ker T = 0$; $\operatorname{ran} T = \mathcal{H}$, $\operatorname{ran} T^* = \mathcal{H}$;
(2) $\ker T^* \neq 0$; $\ker T$, $\ker T^*$ *are finite-dimensional spaces with the same dimension;* $\operatorname{ran} T = (\ker T^*)^\perp$, $\operatorname{ran} T^* = (\ker T)^\perp$.

4.C. Difference quotients and weak derivatives

Difference quotients provide a useful method for proving the weak differentiability of functions. The main result, in Theorem 4.53 below, is that the uniform boundedness of the difference quotients of a function is sufficient to imply that the function is weakly differentiable.

DEFINITION 4.51. If $u : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R} \setminus \{0\}$, the ith difference quotient of u of size h is the function $D_i^h u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

where e_i is the unit vector in the ith direction. The vector of difference quotient is

$$D^h u = \left(D_1^h u, D_2^h u, \dots, D_n^h u\right).$$

The next proposition gives some elementary properties of difference quotients that are analogous to those of derivatives.

PROPOSITION 4.52. *The difference quotient has the following properties.*

(1) *Commutativity with weak derivatives: if $u, \partial_i u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then*
$$\partial_i D_j^h u = D_j^h \partial_i u.$$
(2) *Integration by parts: if $u \in L^p(\mathbb{R}^n)$ and $v \in L^{p'}(\mathbb{R}^n)$, where $1 \leq p \leq \infty$, then*
$$\int (D_i^h u) v \, dx = -\int u (D_i^h v) \, dx.$$
(3) *Product rule:*
$$D_i^h(uv) = u_i^h \left(D_i^h v\right) + \left(D_i^h u\right) v = u \left(D_i^h v\right) + \left(D_i^h u\right) v_i^h.$$
where $u_i^h(x) = u(x + he_i)$.

Proof. Property (1) follows immediately from the linearity of the weak derivative. For (2), note that

$$\begin{aligned}\int (D_i^h u) v \, dx &= \frac{1}{h} \int [u(x+he_i) - u(x)]\, v(x)\, dx \\ &= \frac{1}{h}\int u(x')v(x'-he_i)\, dx' - \frac{1}{h}\int u(x)v(x)\, dx \\ &= \frac{1}{h}\int u(x)\left[v(x-he_i) - v(x)\right]\, dx \\ &= -\int u\left(D_i^{-h} v\right)\, dx.\end{aligned}$$

For (3), we have

$$\begin{aligned}u_i^h\left(D_i^h v\right) + \left(D_i^h u\right) v &= u(x+he_i)\left[\frac{v(x+he_i)-v(x)}{h}\right] + \left[\frac{u(x+he_i)-u(x)}{h}\right] v(x) \\ &= \frac{u(x+he_i)v(x+he_i) - u(x)v(x)}{h} \\ &= D_i^h(uv),\end{aligned}$$

and the same calculation with u and v exchanged. □

Theorem 4.53. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\Omega' \Subset \Omega$. Let*

$$d = \operatorname{dist}(\Omega', \partial\Omega) > 0.$$

(1) *If $Du \in L^p(\Omega)$ where $1 \le p < \infty$, and $0 < |h| < d$, then*

$$\left\|D^h u\right\|_{L^p(\Omega')} \le \|Du\|_{L^p(\Omega)}.$$

(2) *If $u \in L^p(\Omega)$ where $1 < p < \infty$, and there exists a constant C such that*

$$\left\|D^h u\right\|_{L^p(\Omega')} \le C$$

for all $0 < |h| < d/2$, then $u \in W^{1,p}(\Omega')$ and

$$\|Du\|_{L^p(\Omega')} \le C.$$

Proof. To prove (1), we may assume by an approximation argument that u is smooth. Then

$$u(x+he_i) - u(x) = h\int_0^1 \partial_i u(x+te_i)\, dt,$$

and, by Jensen's inequality,

$$|u(x+he_i) - u(x)|^p \le |h|^p \int_0^1 |\partial_i u(x+te_i)|^p\, dt.$$

Integrating this inequality with respect to x, and using Fubini's theorem, together with the fact that $x + te_i \in \Omega$ if $x \in \Omega'$ and $|t| \le h < d$, we get

$$\int_{\Omega'} |u(x+he_i) - u(x)|^p\, dx \le |h|^p \int_\Omega |\partial_i u(x+te_i)|^p\, dx.$$

Thus, $\|D_i^h u\|_{L^p(\Omega')} \le \|D_i^h u\|_{L^p(\Omega)}$, and (1) follows.

To prove (2), note that since

$$\left\{D_i^h u : 0 < |h| < d\right\}$$

is bounded in $L^p(\Omega')$, the Banach-Alaoglu theorem implies that there is a sequence $\{h_k\}$ such that $h_k \to 0$ as $k \to \infty$ and a function $v_i \in L^p(\Omega')$ such that

$$D_i^{h_k} u \rightharpoonup v_i \qquad \text{as } k \to \infty \text{ in } L^p(\Omega').$$

Suppose that $\phi \in C_c^\infty(\Omega')$. Then, for sufficiently small h_k,

$$\int_{\Omega'} u D_i^{-h_k} \phi \, dx = \int_{\Omega'} \left(D_i^{h_k} u \right) \phi \, dx.$$

Taking the limit as $k \to \infty$, when $D_i^{-h_k}\phi$ converges uniformly to $\partial_i \phi$, we get

$$\int_{\Omega'} u \partial_i \phi \, dx = \int_{\Omega'} v_i \phi \, dx.$$

Hence u is weakly differentiable and $\partial_i u = v_i \in L^p(\Omega')$, which proves (2). $\square$