

**Notes on  
Partial Differential Equations**

John K. Hunter

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT DAVIS



# Contents

Chapter 1. Preliminaries	1
1.1. Euclidean space	1
1.2. Spaces of continuous functions	1
1.3. Hölder spaces	2
1.4. $L^p$ spaces	3
1.5. Compactness	6
1.6. Averages	7
1.7. Convolutions	7
1.8. Derivatives and multi-index notation	8
1.9. Mollifiers	10
1.10. Boundaries of open sets	12
1.11. Change of variables	16
1.12. Divergence theorem	16
Chapter 2. Laplace's equation	19
2.1. Mean value theorem	20
2.2. Derivative estimates and analyticity	23
2.3. Maximum principle	26
2.4. Harnack's inequality	31
2.5. Green's identities	32
2.6. Fundamental solution	33
2.7. The Newtonian potential	34
2.8. Singular integral operators	43
Chapter 3. Sobolev spaces	47
3.1. Weak derivatives	47
3.2. Examples	47
3.3. Distributions	50
3.4. Properties of weak derivatives	53
3.5. Sobolev spaces	56
3.6. Approximation of Sobolev functions	57
3.7. Sobolev embedding: $p < n$	57
3.8. Sobolev embedding: $p > n$	66
3.9. Boundary values of Sobolev functions	69
3.10. Compactness results	71
3.11. Sobolev functions on $\Omega \subset \mathbb{R}^n$	73
3.A. Lipschitz functions	75
3.B. Absolutely continuous functions	76
3.C. Functions of bounded variation	78
3.D. Borel measures on $\mathbb{R}$	80

3.E. Radon measures on $\mathbb{R}$	82
3.F. Lebesgue-Stieltjes measures	83
3.G. Integration	84
3.H. Summary	86
Chapter 4. Elliptic PDEs	87
4.1. Weak formulation of the Dirichlet problem	87
4.2. Variational formulation	89
4.3. The space $H^{-1}(\Omega)$	90
4.4. The Poincaré inequality for $H_0^1(\Omega)$	93
4.5. Existence of weak solutions of the Dirichlet problem	94
4.6. General linear, second order elliptic PDEs	96
4.7. The Lax-Milgram theorem and general elliptic PDEs	98
4.8. Compactness of the resolvent	101
4.9. The Fredholm alternative	102
4.10. The spectrum of a self-adjoint elliptic operator	104
4.11. Interior regularity	105
4.12. Boundary regularity	109
4.13. Some further perspectives	112
4.A. Heat flow	114
4.B. Operators on Hilbert spaces	116
4.C. Difference quotients and weak derivatives	119
Chapter 5. The Heat Equation	123
5.1. The initial value problem	123
5.A. The Schwartz space and the Fourier transform	132
Bibliography	139