Notes on
Partial Differential Equations

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Contents

Chapter 1. Preliminaries 1
1.1. Euclidean space 1
1.2. Spaces of continuous functions 1
1.3. Hölder spaces 2
1.4. L^p spaces 3
1.5. Compactness 6
1.6. Averages 7
1.7. Convolutions 7
1.8. Derivatives and multi-index notation 8
1.9. Mollifiers 10
1.10. Boundaries of open sets 12
1.11. Change of variables 16
1.12. Divergence theorem 16

Chapter 2. Laplace's equation 19
2.1. Mean value theorem 20
2.2. Derivative estimates and analyticity 23
2.3. Maximum principle 26
2.4. Harnack's inequality 31
2.5. Green's identities 32
2.6. Fundamental solution 33
2.7. The Newtonian potential 34
2.8. Singular integral operators 43

Chapter 3. Sobolev spaces 47
3.1. Weak derivatives 47
3.2. Examples 47
3.3. Distributions 50
3.4. Properties of weak derivatives 53
3.5. Sobolev spaces 56
3.6. Approximation of Sobolev functions 57
3.7. Sobolev embedding: $p < n$ 57
3.8. Sobolev embedding: $p > n$ 66
3.9. Boundary values of Sobolev functions 69
3.10. Compactness results 71
3.11. Sobolev functions on $\Omega \subset \mathbb{R}^n$ 73
3.A. Lipschitz functions 75
3.B. Absolutely continuous functions 76
3.C. Functions of bounded variation 78
3.D. Borel measures on $\mathbb{R}$ 80

3.E. Radon measures on $\mathbb{R}$ 82
3.F. Lebesgue-Stieltjes measures 83
3.G. Integration 84
3.H. Summary 86

Chapter 4. Elliptic PDEs 87
4.1. Weak formulation of the Dirichlet problem 87
4.2. Variational formulation 89
4.3. The space $H^{-1}(\Omega)$ 90
4.4. The Poincaré inequality for $H^1_0(\Omega)$ 93
4.5. Existence of weak solutions of the Dirichlet problem 94
4.6. General linear, second order elliptic PDEs 96
4.7. The Lax-Milgram theorem and general elliptic PDEs 98
4.8. Compactness of the resolvent 101
4.9. The Fredholm alternative 102
4.10. The spectrum of a self-adjoint elliptic operator 104
4.11. Interior regularity 105
4.12. Boundary regularity 109
4.13. Some further perspectives 112
4.A. Heat flow 114
4.B. Operators on Hilbert spaces 116
4.C. Difference quotients and weak derivatives 119

Chapter 5. The Heat Equation 123
5.1. The initial value problem 123
5.A. The Schwartz space and the Fourier transform 132

Bibliography 139

CHAPTER 1

Preliminaries

In this chapter, we collect various definitions and theorems for future use. Proofs may be found in the references e.g. [**7, 13, 16, 17, 18**].

1.1. Euclidean space

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. We denote the Euclidean norm of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ by

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

and the inner product of vectors $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We denote Lebesgue measure on $\mathbb{R}^n$ by dx, and the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$.

If E is a subset of $\mathbb{R}^n$, we denote the complement by $E^c = \mathbb{R}^n \setminus E$, the closure by $\overline{E}$, the interior by E° and the boundary by $\partial E = \overline{E} \setminus E^\circ$. The characteristic function $\chi_E : \mathbb{R}^n \to \mathbb{R}$ of E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

A set E is bounded if $\{|x| : x \in E\}$ is bounded in $\mathbb{R}$. A set is connected if it is not the disjoint union of two nonempty relatively open subsets. We sometimes refer to a connected open set as a domain.

We say that an open set Ω' in $\mathbb{R}^n$ is compactly contained in an open set Ω, written $\Omega' \Subset \Omega$, if $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact. If $\overline{\Omega'} \subset \Omega$, then

$$\operatorname{dist}(\Omega', \partial\Omega) = \inf\{|x - y| : x \in \Omega', y \in \partial\Omega\} > 0.$$

This distance is finite provided that $\Omega' \neq \emptyset$ and $\Omega \neq \mathbb{R}^n$.

1.2. Spaces of continuous functions

Let Ω be an open set in $\mathbb{R}^n$. We denote the space of continuous functions $u : \Omega \to \mathbb{R}$ by $C(\Omega)$; the space of functions with continuous partial derivatives in Ω of order less than or equal to $k \in \mathbb{N}$ by $C^k(\Omega)$; and the space of functions with continuous derivatives of all orders by $C^\infty(\Omega)$. Functions in these spaces need not be bounded even if Ω is bounded; for example, $(1/x) \in C^\infty(0, 1)$.

If Ω is a bounded open set in $\mathbb{R}^n$, we denote by $C(\overline{\Omega})$ the space of continuous functions $u : \overline{\Omega} \to \mathbb{R}$. This is a Banach space with respect to the maximum, or supremum, norm

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|.$$

We denote the support of a continuous function $u : \Omega \to \mathbb{R}^n$ by

$$\operatorname{spt} u = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We denote by $C_c(\Omega)$ the space of continuous functions whose support is compactly contained in Ω, and by $C_c^\infty(\Omega)$ the space of functions with continuous derivatives of all orders and compact support in Ω. We will sometimes refer to such functions as test functions.

The completion of $C_c(\mathbb{R}^n)$ with respect to the uniform norm is the space $C_0(\mathbb{R}^n)$ of continuous functions that approach zero at infinity. (Note that in many places the notation C_0 and C_0^∞ is used to denote the spaces of compactly supported functions that we denote by C_c and C_c^∞.)

If Ω is bounded, we say that a function $u : \overline{\Omega} \to \mathbb{R}$ belongs to $C^k(\overline{\Omega})$ if it is continuous and its partial derivatives of order less than or equal to k are uniformly continuous in Ω, in which case they extend to continuous functions on $\overline{\Omega}$. The space $C^k(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq k} \sup_{\Omega} |\partial^\alpha u|$$

where we use the multi-index notation for partial derivatives explained in Section 1.8. This norm is finite because the derivatives $\partial^\alpha u$ are continuous functions on the compact set $\overline{\Omega}$.

A vector field $X : \Omega \to \mathbb{R}^m$ belongs to $C^k(\overline{\Omega})$ if each of its components belongs to $C^k(\overline{\Omega})$.

1.3. Hölder spaces

The definition of continuity is not a quantitative one, because it does not say how rapidly the values $u(y)$ of a function approach its value $u(x)$ as $y \to x$. The modulus of continuity $\omega : [0, \infty] \to [0, \infty]$ of a general continuous function u, satisfying

$$|u(x) - u(y)| \leq \omega\left(|x - y|\right),$$

may decrease arbitrarily slowly. As a result, despite their simple and natural appearance, spaces of continuous functions are often not suitable for the analysis of PDEs, which is almost always based on quantitative estimates.

A straightforward and useful way to strengthen the definition of continuity is to require that the modulus of continuity is proportional to a power $|x - y|^\alpha$ for some exponent $0 < \alpha \leq 1$. Such functions are said to be Hölder continuous, or Lipschitz continuous if $\alpha = 1$. Roughly speaking, one can think of Hölder continuous functions with exponent α as functions with bounded fractional derivatives of the the order α.

Definition 1.1. Suppose that Ω is an open set in $\mathbb{R}^n$ and $0 < \alpha \leq 1$. A function $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent α in Ω if the quantity

$$[u]_{\alpha,\Omega} = \sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \tag{1.1}$$

is finite. A function $u : \Omega \to \mathbb{R}$ is locally uniformly Hölder continuous with exponent α in Ω if $[u]_{\alpha,\Omega'}$ is finite for every $\Omega' \Subset \Omega$. We denote by $C^{0,\alpha}(\Omega)$ the space of locally uniformly Hölder continuous functions with exponent α in Ω. If Ω is bounded, we denote by $C^{0,\alpha}\left(\overline{\Omega}\right)$ the space of uniformly Hölder continuous functions with exponent α in Ω.

We typically use Greek letters such as α, β both for Hölder exponents and multi-indices; it should be clear from the context which they denote.

When α and Ω are understood, we will abbreviate 'u is (locally) uniformly Hölder continuous with exponent α in Ω' to 'u is (locally) Hölder continuous.' If u is Hölder continuous with exponent one, then we say that u is Lipschitz continuous. There is no purpose in considering Hölder continuous functions with exponent greater than one, since any such function is differentiable with zero derivative, and is therefore constant.

The quantity $[u]_{\alpha,\Omega}$ is a semi-norm, but it is not a norm since it is zero for constant functions. The space $C^{0,\alpha}\left(\overline{\Omega}\right)$, where Ω is bounded, is a Banach space with respect to the norm

$$\|u\|_{C^{0,\alpha}\left(\overline{\Omega}\right)} = \sup_{\Omega} |u| + [u]_{\alpha,\Omega}\,.$$

EXAMPLE 1.2. For $0 < \alpha < 1$, define $u(x) : (0,1) \to \mathbb{R}$ by $u(x) = |x|^{\alpha}$. Then $u \in C^{0,\alpha}\left([0,1]\right)$, but $u \notin C^{0,\beta}\left([0,1]\right)$ for $\alpha < \beta \le 1$.

EXAMPLE 1.3. The function $u(x) : (-1,1) \to \mathbb{R}$ given by $u(x) = |x|$ is Lipschitz continuous, but not continuously differentiable. Thus, $u \in C^{0,1}\left([-1,1]\right)$, but $u \notin C^{1}\left([-1,1]\right)$.

We may also define spaces of continuously differentiable functions whose kth derivative is Hölder continuous.

DEFINITION 1.4. If Ω is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $0 < \alpha \le 1$, then $C^{k,\alpha}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$ with continuous partial derivatives in Ω of order less than or equal to k whose kth partial derivatives are locally uniformly Hölder continuous with exponent α in Ω. If the open set Ω is bounded, then $C^{k,\alpha}\left(\overline{\Omega}\right)$ consists of functions with uniformly continuous partial derivatives in Ω of order less than or equal to k whose kth partial derivatives are uniformly Hölder continuous with exponent α in Ω.

The space $C^{k,\alpha}\left(\overline{\Omega}\right)$ is a Banach space with respect to the norm

$$\|u\|_{C^{k,\alpha}\left(\overline{\Omega}\right)} = \sum_{|\beta|\le k} \sup_{\Omega} \left|\partial^{\beta} u\right| + \sum_{|\beta|=k} \left[\partial^{\beta} u\right]_{\alpha,\Omega}$$

1.4. L^p spaces

As before, let Ω be an open set in $\mathbb{R}^n$ (or, more generally, a Lebesgue-measurable set).

DEFINITION 1.5. For $1 \le p < \infty$, the space $L^p(\Omega)$ consists of the Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |f|^p \, dx < \infty,$$

and $L^{\infty}(\Omega)$ consists of the essentially bounded functions.

These spaces are Banach spaces with respect to the norms

$$\|f\|_p = \left(\int_\Omega |f|^p \, dx\right)^{1/p}, \qquad \|f\|_\infty = \sup_\Omega |f|$$

where sup denotes the essential supremum,

$$\sup_\Omega f = \inf\left\{M \in \mathbb{R} : f \leq M \text{ almost everywhere in } \Omega\right\}.$$

Strictly speaking, elements of the Banach space L^p are equivalence classes of functions that are equal almost everywhere, but we identify a function with its equivalence class unless we need to refer to the pointwise values of a specific representative. For example, we say that a function $f \in L^p(\Omega)$ is continuous if it is equal almost everywhere to a continuous function, and that it has compact support if it is equal almost everywhere to a function with compact support.

Next we summarize some fundamental inequalities for integrals, in addition to Minkowski's inequality which is implicit in the statement that $\|\cdot\|_{L^p}$ is a norm for $p \geq 1$.

Jensen's inequality states that the value of a convex function at a mean is less than or equal to the mean of the values of the convex function.

Theorem 1.6. *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, Ω is a set in $\mathbb{R}^n$ with finite Lebesgue measure, and $f \in L^1(\Omega)$. Then*

$$\phi\left(\frac{1}{|\Omega|}\int_\Omega f \, dx\right) \leq \frac{1}{|\Omega|}\int_\Omega \phi \circ f \, dx.$$

To state the next inequality, we first define the Hölder conjugate of an exponent p. We denote it by p' to distinguish it from the Sobolev conjugate p^* which we will introduce later on.

Definition 1.7. The Hölder conjugate of $p \in [1, \infty]$ is the quantity $p' \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the convention that $1/\infty = 0$.

The following result is called Hölder's inequality.[1] The special case when $p = p' = 1/2$ is the Cauchy-Schwartz inequality.

Theorem 1.8. *If $1 \leq p \leq \infty$, $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

Repeated application of this inequality gives the following generalization.

Theorem 1.9. *If $1 \leq p_i \leq \infty$ for $1 \leq i \leq N$ satisfy*

$$\sum_{i=1}^N \frac{1}{p_i} = 1$$

[1] In retrospect, it would've been better to use $L^{1/p}$ spaces instead of L^p spaces, just as it would've been better to use inverse temperature instead of temperature, with absolute zero corresponding to infinite coldness.

and $f_i \in L^{p_i}(\Omega)$ *for* $1 \le i \le N$*, then* $f = \prod_{i=1}^N f_i \in L^1(\Omega)$ *and*

$$\|f\|_1 \le \prod_{i=1}^N \|f_i\|_{p_i}.$$

If Ω has finite measure and $1 \le q \le p$, then Hölder's inequality shows that $f \in L^p(\Omega)$ implies that $f \in L^q(\Omega)$ and

$$\|f\|_q \le |\Omega|^{1/q-1/p}\|f\|_p.$$

Thus, the embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. This result is not true if the measure of Ω is infinite, but in general we have the following interpolation result.

LEMMA 1.10. *If* $p \le q \le r$*, then* $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega)$ *and*

$$\|f\|_q \le \|f\|_p^\theta \|f\|_r^{1-\theta}$$

where $0 \le \theta \le 1$ *is given by*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

PROOF. Assume without loss of generality that $f \ge 0$. Using Hölder's inequality with exponents $1/\sigma$ and $1/(1-\sigma)$, we get

$$\int f^q\,dx = \int f^{\theta q} f^{(1-\theta)q}\,dx \le \left(\int f^{\theta q/\sigma}\,dx\right)^\sigma \left(\int f^{(1-\theta)q/(1-\sigma)}\,dx\right)^{1-\sigma}.$$

Choosing $\sigma/\theta = q/p$, when $(1-\sigma)/(1-\theta) = q/r$, we get

$$\int f^q\,dx \le \left(\int f^p\,dx\right)^{q\theta/p} \left(\int f^r\,dx\right)^{q(1-\theta)/r}$$

and the result follows. □

It is often useful to consider local L^p spaces consisting of functions that have finite integral on compact sets.

DEFINITION 1.11. The space $L^p_{\mathrm{loc}}(\Omega)$, where $1 \le p \le \infty$, consists of functions $f : \Omega \to \mathbb{R}$ such that $f \in L^p(\Omega')$ for every open set $\Omega' \Subset \Omega$. A sequence of functions $\{f_n\}$ converges to f in $L^p_{\mathrm{loc}}(\Omega)$ if $\{f_n\}$ converges to f in $L^p(\Omega')$ for every open set $\Omega' \Subset \Omega$.

If $p < q$, then $L^q_{\mathrm{loc}}(\Omega) \hookrightarrow L^p_{\mathrm{loc}}(\Omega)$ even if the measure of Ω is infinite. Thus, $L^1_{\mathrm{loc}}(\Omega)$ is the 'largest' space of integrable functions on Ω.

EXAMPLE 1.12. Consider $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|^a}$$

where $a \in \mathbb{R}$. Then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $a < n$. To prove this, let

$$f^\epsilon(x) = \begin{cases} f(x) & \text{if } |x| > \epsilon, \\ 0 & \text{if } |x| \le \epsilon. \end{cases}$$

Then $\{f^\epsilon\}$ is monotone increasing and converges pointwise almost everywhere to f as $\epsilon \to 0^+$. For any $R > 0$, the monotone convergence theorem implies that

$$\begin{aligned}\int_{B_R(0)} f\, dx &= \lim_{\epsilon \to 0^+} \int_{B_R(0)} f^\epsilon\, dx \\ &= \lim_{\epsilon \to 0^+} \int_\epsilon^R r^{n-a-1}\, dr \\ &= \begin{cases} \infty & \text{if } n - a \leq 0, \\ (n-a)^{-1} R^{n-a} & \text{if } n - a > 0, \end{cases}\end{aligned}$$

which proves the result. The function f does not belong to $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ for any value of a, since the integral of f^p diverges at infinity whenever it converges at zero.

1.5. Compactness

A subset F of a metric space X is precompact if the closure of F is compact; equivalently, F is precompact if every sequence in F has a subsequence that converges in X.

The Arzelà-Ascoli theorem gives a basic criterion for compactness in function spaces: it states that a set of continuous functions on a compact metric space is precompact if and only if it is bounded and equicontinuous. We state the result explicitly for the spaces of interest here.

THEOREM 1.13. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$. A subset $\mathcal{F}$ of $C\left(\overline{\Omega}\right)$, equipped with the maximum norm, is precompact if and only if:*

(1) *there exists a constant M such that*
$$\|f\|_\infty \leq M \qquad \text{for all } f \in \mathcal{F};$$
(2) *for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, x + h \in \overline{\Omega}$ and $|h| < \delta$ then*
$$|f(x+h) - f(x)| < \epsilon \qquad \text{for all } f \in \mathcal{F}.$$

The following theorem (known as the Fréchet-Kolmogorov, Kolmogorov-Riesz, or Riesz-Tamarkin theorem) gives conditions analogous to the ones in the Arzelà-Ascoli theorem for a set to be precompact in $L^p(\mathbb{R})$, namely that the set is bounded, 'tight', and L^p-equicontinuous.

THEOREM 1.14. *Let $1 \leq p < \infty$. A subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$ is precompact if and only if:*

(1) *there exists M such that*
$$\|f\|_{L^p} \leq M \qquad \text{for all } f \in \mathcal{F};$$
(2) *for every $\epsilon > 0$ there exists R such that*
$$\left(\int_{|x|>R} |f(x)|^p\, dx\right)^{1/p} < \epsilon \qquad \text{for all } f \in \mathcal{F}.$$
(3) *for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$,*
$$\left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p\, dx\right)^{1/p} < \epsilon \qquad \text{for all } f \in \mathcal{F}.$$

For a proof, see [**18**].

1.6. Averages

For $x \in \mathbb{R}^n$ and $r > 0$, let

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$

denote the open ball centered at x with radius r, and

$$\partial B_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}$$

the corresponding sphere.

The volume of the unit ball in $\mathbb{R}^n$ is given by

$$\alpha_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$$

where Γ is the Gamma function, which satisfies

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(1) = 1, \qquad \Gamma(x+1) = x\Gamma(x).$$

Thus, for example, $\alpha_2 = \pi$ and $\alpha_3 = 4\pi/3$. An integration with respect to polar coordinates shows that the area of the $(n-1)$-dimensional unit sphere is $n\alpha_n$.

We denote the average of a function $f \in L^1_{\text{loc}}(\Omega)$ over a ball $B_r(x) \Subset \Omega$, or the corresponding sphere $\partial B_r(x)$, by

$$\fint_{B_r(x)} f\,dx = \frac{1}{\alpha_n r^n}\int_{B_r(x)} f\,dx, \qquad \fint_{\partial B_r(x)} f\,dS = \frac{1}{n\alpha_n r^{n-1}}\int_{\partial B_r(x)} f\,dS. \tag{1.2}$$

If f is continuous at x, then

$$\lim_{r\to 0^+}\fint_{B_r(x)} f\,dx = f(x).$$

The following result, called the Lebesgue differentiation theorem, implies that the averages of a locally integrable function converge pointwise almost everywhere to the function as the radius r shrinks to zero.

THEOREM 1.15. *If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ then*

$$\lim_{r\to 0^+}\fint_{B_r(x)} |f(y) - f(x)|\,dx = 0 \tag{1.3}$$

pointwise almost everywhere for $x \in \mathbb{R}^n$.

A point $x \in \mathbb{R}^n$ for which (1.3) holds is called a Lebesgue point of f. For a proof of this theorem (using the Wiener covering lemma and the Hardy-Littlewood maximal function) see Folland [**7**] or Taylor [**17**].

1.7. Convolutions

DEFINITION 1.16. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are measurable function, we define the convolution $f * g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\,dy$$

provided that the integral converges pointwise almost everywhere in x.

When defined, the convolutiom product is both commutative and associative,

$$f * g = g * f, \qquad f * (g * h) = (f * g) * h.$$

In many respects, the convolution of two functions inherits the best properties of both functions.

If $f, g \in C_c(\mathbb{R}^n)$, then their convolution also belongs to $C_c(\mathbb{R}^n)$ and

$$\operatorname{spt}(f * g) \subset \operatorname{spt} f + \operatorname{spt} g.$$

If $f \in C_c(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n)$, then $f*g \in C(\mathbb{R}^n)$ is defined, however rapidly g grows at infinity, but typically it does not have compact support. If neither f nor g have compact support, we need some conditions on their growth or decay at infinity to ensure that the convolution exists. The following result, called Young's inequality, gives conditions for the convolution of L^p functions to exist and estimates its norm.

Theorem 1.17. *Suppose that* $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

If $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

The following special cases are useful to keep in mind.

Example 1.18. If $p = q = 2$ then $r = \infty$. In this case, the result follows from the Cauchy-Schwartz inequality, since for any $x \in \mathbb{R}^n$

$$\left| \int f(x-y) g(y)\, dx \right| \le \|f\|_{L^2} \|g\|_{L^2}.$$

Moreover, a density argument shows that $f * g \in C_0(\mathbb{R}^n)$: Choose $f_k, g_k \in C_c(\mathbb{R}^n)$ such that $f_k \to f$, $g_k \to g$ in $L^2(\mathbb{R}^n)$, then $f_k * g_k \in C_c(\mathbb{R}^n)$ and $f_k * g_k \to f * g$ uniformly. A similar argument is used in the proof of the Riemann-Lebesgue lemma that $\hat{f} \in C_0(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$.

Example 1.19. If $p = q = 1$, then $r = 1$, and the result follows directly from Fubini's theorem, since

$$\int \left| \int f(x-y) g(y)\, dy \right| dx \le \int |f(x-y) g(y)|\, dx dy = \left(\int |f(x)|\, dx \right) \left(\int |g(y)|\, dy \right).$$

Thus, the space $L^1(\mathbb{R}^n)$ is an algebra under the convolution product. The Fourier transform maps the convolution product of two L^1-functions to the pointwise product of their Fourier transforms.

Example 1.20. If $q = 1$, then $p = r$. Thus convolution with an integrable function $k \in L^1(\mathbb{R}^n)$, is a bounded linear map $f \mapsto k * f$ on $L^p(\mathbb{R}^n)$.

1.8. Derivatives and multi-index notation

We define the derivative of a scalar field $u : \Omega \to \mathbb{R}$ by

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right).$$

We will also denote the ith partial derivative by $\partial_i u$, the ijth derivative by $\partial_{ij} u$, and so on. The divergence of a vector field $X = (X_1, X_2, \dots, X_n) : \Omega \to \mathbb{R}^n$ is

$$\operatorname{div} X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \dots + \frac{\partial X_n}{\partial x_n}.$$

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denote the non-negative integers. An n-dimensional multi-indexis a vector $\alpha \in \mathbb{N}_0^n$, meaning that

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \qquad \alpha_i = 0, 1, 2, \dots.$$

We write

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \qquad \alpha! = \alpha_1!\alpha_2! \dots \alpha_n!.$$

We define derivatives and powers of order α by

$$\partial^\alpha = \frac{\partial}{\partial x^{\alpha_1}} \frac{\partial}{\partial x^{\alpha_2}} \dots \frac{\partial}{\partial x^{\alpha_n}}, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are multi-indices, we define the multi-index $(\alpha + \beta)$ by

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n).$$

We denote by $\chi_n(k)$ the number of multi-indices $\alpha \in \mathbb{N}_0^n$ with order $0 \le |\alpha| \le k$, and by $\tilde{\chi}_n(k)$ the number of multi-indices with order $|\alpha| = k$. Then

$$\chi_n(k) = \frac{(n+k)!}{n!k!}, \qquad \tilde{\chi}_n(k) = \frac{(n+k-1)!}{(n-1)!k!}$$

1.8.1. Taylor's theorem for functions of several variables. The multi-index notation provides a compact way to write the multinomial theorem and the Taylor expansion of a function of several variables. The multinomial expansion of a power is

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{\alpha_1 + \dots \alpha_n = k} \binom{k}{\alpha_1 \alpha_2 \dots \alpha_n} x_i^{\alpha_i} = \sum_{|\alpha| = k} \binom{k}{\alpha} x^\alpha$$

where the multinomial coefficient of a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of order $|\alpha| = k$ is given by

$$\binom{k}{\alpha} = \binom{k}{\alpha_1 \alpha_2 \dots \alpha_n} = \frac{k!}{\alpha_1!\alpha_2! \dots \alpha_n!}.$$

THEOREM 1.21. *Suppose that $u \in C^k(B_r(x))$ and $h \in B_r(0)$. Then*

$$u(x+h) = \sum_{|\alpha| \le k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x, h)$$

where the remainder is given by

$$R_k(x, h) = \sum_{|\alpha| = k} \frac{\partial^\alpha u(x + \theta h)}{\alpha!} h^\alpha$$

for some $0 < \theta < 1$.

PROOF. Let $f(t) = u(x + th)$ for $0 \leq t \leq 1$. Taylor's theorem for a function of a single variable implies that

$$f(1) = \sum_{j=0}^{k-1} \frac{1}{j!}\frac{d^j f}{dt^j}(0) + \frac{1}{k!}\frac{d^k f}{dt^k}(\theta)$$

for some $0 < \theta < 1$. By the chain rule,

$$\frac{df}{dt} = Du \cdot h = \sum_{i=1}^{n} h_i \partial_i u,$$

and the multinomial theorem gives

$$\frac{d^k}{dt^k} = \left(\sum_{i=1}^{n} h_i \partial_i\right)^k = \sum_{|\alpha|=k} \binom{n}{\alpha} h^\alpha \partial^\alpha.$$

Using this expression to rewrite the Taylor series for f in terms of u, we get the result. □

A function $u : \Omega \to \mathbb{R}$ is real-analytic in an open set Ω if it has a power-series expansion that converges to the function in a ball of non-zero radius about every point of its domain. We denote by $C^\omega(\Omega)$ the space of real-analytic functions on Ω. A real-analytic function is C^∞, since its Taylor series can be differentiated term-by-term, but a C^∞ function need not be real-analytic. For example, see (1.4) below.

1.9. Mollifiers

The function

$$\eta(x) = \begin{cases} C \exp\left[-1/(1-|x|^2)\right] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{1.4}$$

belongs to $C_c^\infty(\mathbb{R}^n)$ for any constant C. We choose C so that

$$\int_{\mathbb{R}^n} \eta \, dx = 1$$

and for any $\epsilon > 0$ define the function

$$\eta^\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right). \tag{1.5}$$

Then η^ϵ is a C^∞-function with integral equal to one whose support is the closed ball $\overline{B}_\epsilon(0)$. We refer to (1.5) as the 'standard mollifier.'

We remark that $\eta(x)$ in (1.4) is not real-analytic when $|x| = 1$. All of its derivatives are zero at those points, so the Taylor series converges to zero in any neighborhood, not to the original function. The only function that is real-analytic with compact support is the zero function. In rough terms, an analytic function is a single 'organic' entity: its values in, for example, a single open ball determine its values everywhere in a maximal domain of analyticity (which is a Riemann surface) through analytic continuation. The behavior of C^∞-function at one point is, however, completely unrelated to its behavior at another point.

Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$ is a locally integrable function. For $\epsilon > 0$, let

$$\Omega^\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\} \tag{1.6}$$

and define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ by

$$f^\epsilon(x) = \int_\Omega \eta^\epsilon(x-y) f(y)\, dy \tag{1.7}$$

where η^ϵ is the mollifier in (1.5). We define f^ϵ for $x \in \Omega^\epsilon$ so that $B_\epsilon(x) \subset \Omega$ and we have room to average f. If $\Omega = \mathbb{R}^n$, we have simply $\Omega^\epsilon = \mathbb{R}^n$. The function f^ϵ is a smooth approximation of f.

THEOREM 1.22. *Suppose that $f \in L^p_{\mathrm{loc}}(\Omega)$ for $1 \le p < \infty$, and $\epsilon > 0$. Define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ by (1.7). Then: (a) $f^\epsilon \in C^\infty(\Omega^\epsilon)$ is smooth; (b) $f^\epsilon \to f$ pointwise almost everywhere in Ω as $\epsilon \to 0^+$; (c) $f^\epsilon \to f$ in $L^p_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0^+$.*

PROOF. The smoothness of f^ϵ follows by differentiation under the integral sign

$$\partial^\alpha f^\epsilon(x) = \int_\Omega \partial^\alpha \eta^\epsilon(x-y) f(y)\, dy$$

which may be justified by use of the dominated convergence theorem. The pointwise almost everywhere convergence (at every Lebesgue point of f) follows from the Lebesgue differentiation theorem. The convergence in L^p_{loc} follows by the approximation of f by a continuous function (for which the result is easy to prove) and the use of Young's inequality, since $\|\eta^\epsilon\|_{L^1} = 1$ is bounded independently of ϵ. □

One consequence of this theorem is that the space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$. Note that this is not true when $p = \infty$, since the uniform limit of smooth test functions is continuous.

1.9.1. Cutoff functions.

THEOREM 1.23. *Suppose that $\Omega' \Subset \Omega$ are open sets in $\mathbb{R}^n$. Then there is a function $\phi \in C_c^\infty(\Omega)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on Ω'.*

PROOF. Let $d = \operatorname{dist}(\Omega', \partial\Omega)$ and define

$$\Omega'' = \{x \in \Omega : \operatorname{dist}(x, \Omega') < d/2\}.$$

Let χ be the characteristic function of Ω'', and define $\phi = \eta^{d/2} * \chi$ where η^ϵ is the standard mollifier. Then one may verify that ϕ has the required properties. □

We refer to a function with the properties in this theorem as a cutoff function.

EXAMPLE 1.24. If $0 < r < R$ and $\Omega'' = B_r(0)$, $\Omega' = B_R(0)$ are balls in $\mathbb{R}^n$, then the corresponding cut-off function ϕ satisfies

$$|D\phi| \le \frac{C}{R-r}$$

where C is a constant that is independent of r, R.

1.9.2. Partitions of unity. Partitions of unity allow us to piece together global results from local results.

THEOREM 1.25. *Suppose that K is a compact set in $\mathbb{R}^n$ which is covered by a finite collection $\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ of open sets. Then there exists a collection of functions $\{\eta_1, \eta_2, \ldots, \eta_N\}$ such that $0 \le \eta_i \le 1$, $\eta_i \in C_c^\infty(\Omega_i)$, and $\sum_{i=1}^N \eta_i = 1$ on K.*

We call $\{\eta_i\}$ a partition of unity subordinate to the cover $\{\Omega_i\}$. To prove this result, we use Urysohn's lemma to construct a collection of continuous functions with the desired properties, then use mollification to obtain a collection of smooth functions.

1.10. Boundaries of open sets

When we analyze solutions of a PDE in the interior of their domain of definition, we can often consider domains that are arbitrary open sets and analyze the solutions in a sufficiently small ball. In order to analyze the behavior of solutions at a boundary, however, we typically need to assume that the boundary has some sort of smoothness. In this section, we define the smoothness of the boundary of an open set. We also explain briefly how one defines analytically the normal vector-field and the surface area measure on a smooth boundary.

In general, the boundary of an open set may be complicated. For example, it can have nonzero Lebesgue measure.

EXAMPLE 1.26. Let $\{q_i : i \in \mathbb{N}\}$ be an enumeration of the rational numbers $q_i \in (0,1)$. For each $i \in \mathbb{N}$, choose an open interval $(a_i, b_i) \subset (0,1)$ that contains q_i, and let

$$\Omega = \bigcup_{i\in\mathbb{N}} (a_i, b_i).$$

The Lebesgue measure of $|\Omega| > 0$ is positive, but we can make it as small as we wish; for example, choosing $b_i - a_i = \epsilon 2^{-i}$, we get $|\Omega| \leq \epsilon$. One can check that $\partial\Omega = [0,1] \setminus \Omega$. Thus, if $|\Omega| < 1$, then $\partial\Omega$ has nonzero Lebesgue measure.

Moreover, an open set, or domain, need not lie on one side of its boundary (we say that Ω lies on one side of its boundary if $\overline{\Omega}^\circ = \Omega$), and corners, cusps, or other singularities in the boundary cause analytical difficulties.

EXAMPLE 1.27. The unit disc in $\mathbb{R}^2$ with the nonnegative x-axis removed,

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\} \setminus \left\{(x,0) \in \mathbb{R}^2 : 0 \leq x < 1\right\},$$

does not lie on one side of its boundary.

In rough terms, the boundary of an open set is smooth if it can be 'flattened out' locally by a smooth map.

DEFINITION 1.28. Suppose that $k \in \mathbb{N}$. A map $\phi : U \to V$ between open sets U, V in $\mathbb{R}^n$ is a C^k-diffeomorphism if it one-to-one, onto, and ϕ and ϕ^{-1} have continuous derivatives of order less than or equal to k.

Note that the derivative $D\phi(x) : \mathbb{R}^n \to \mathbb{R}^n$ of a diffeomorphism $\phi : U \to V$ is an invertible linear map for every $x \in U$.

DEFINITION 1.29. Let Ω be a bounded open set in $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that the boundary $\partial\Omega$ is C^k, or that Ω is C^k for short, if for every $x \in \overline{\Omega}$ there is an open neighborhood $U \subset \mathbb{R}^n$ of x, an open set $V \subset \mathbb{R}^n$, and a C^k-diffeomorphism $\phi : U \to V$ such that

$$\phi(U \cap \Omega) = V \cap \{y_n > 0\}, \qquad \phi(U \cap \partial\Omega) = V \cap \{y_n = 0\}$$

where $(y_1, \ldots, y_n)$ are coordinates in the image space $\mathbb{R}^n$.

If ϕ is a C^∞-diffeomorphism, then we say that the boundary is C^∞, with an analogous definition of a Lipschitz or analytic boundary.

In other words, the definition says that a C^k open set in $\mathbb{R}^n$ is an n-dimensional C^k-manifold with boundary. The maps ϕ in Definition 1.29 are coordinate charts for the manifold. It follows from the definition that Ω lies on one side of its boundary and that $\partial\Omega$ is an oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ without boundary. The standard orientation is given by the outward-pointing normal (see below).

Example 1.30. The open set

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : x > 0,\, y > \sin(1/x)\right\}$$

lies on one side of its boundary, but the boundary is not C^1 since there is no coordinate chart of the required form for the boundary points $\{(x, 0) : -1 \le x \le 1\}$.

1.10.1. Open sets in the plane. A simple closed curve, or Jordan curve, Γ is a set in the plane that is homeomorphic to a circle. That is, $\Gamma = \gamma(\mathbb{T})$ is the image of a one-to-one continuous map $\gamma : \mathbb{T} \to \mathbb{R}^2$ with continuous inverse $\gamma^{-1} : \Gamma \to \mathbb{T}$. (The requirement that the inverse is continuous follows from the other assumptions.) According to the Jordan curve theorem, a Jordan curve divides the plane into two disjoint connected open sets, so that $\mathbb{R}^2 \setminus \Gamma = \Omega_1 \cup \Omega_2$. One of the sets (the 'interior') is bounded and simply connected. The interior region of a Jordan curve is called a Jordan domain.

Example 1.31. The slit disc Ω in Example 1.27 is not a Jordan domain. For example, its boundary separates into three nonempty connected components when the point $(1, 0)$ is removed, but the circle remains connected when any point is removed, so $\partial\Omega$ cannot be homeomorphic to the circle.

Example 1.32. The interior Ω of the Koch, or 'snowflake,' curve is a Jordan domain. The Hausdorff dimension of its boundary is strictly greater than one. It is interesting to note that, despite the irregular nature of its boundary, this domain has the property that every function in $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $1 \le p < \infty$ can be extended to a function in $W^{k,p}(\mathbb{R}^2)$.

If $\gamma : \mathbb{T} \to \mathbb{R}^2$ is one-to-one, C^1, and $|D\gamma| \neq 0$, then the image of γ is the C^1 boundary of the open set which it encloses. The condition that γ is one-to-one is necessary to avoid self-intersections (for example, a figure-eight curve), and the condition that $|D\gamma| \neq 0$ is necessary in order to ensure that the image is a C^1-submanifold of $\mathbb{R}^2$.

Example 1.33. The curve $\gamma : t \mapsto \left(t^2, t^3\right)$ is not C^1 at $t = 0$ where $D\gamma(0) = 0$.

1.10.2. Parametric representation of a boundary. If Ω is an open set in $\mathbb{R}^n$ with C^k-boundary and ϕ is a chart on a neighborhood U of a boundary point, as in Definition 1.29, then we can define a local chart

$$\Phi = (\Phi_1, \Phi_2, \dots, \Phi_{n-1}) : U \cap \partial\Omega \subset \mathbb{R}^n \to W \subset \mathbb{R}^{n-1}$$

for the boundary $\partial\Omega$ by $\Phi = (\phi_1, \phi_2, \dots, \phi_{n-1})$. Thus, $\partial\Omega$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

The boundary is parametrized locally by $x_i = \Psi_i\,(y_1, y_2, \dots, y_{n-1})$ where $1 \le i \le n$ and $\Psi = \Phi^{-1} : W \to U \cap \partial\Omega$. The $(n-1)$-dimensional tangent space of $\partial\Omega$ is spanned by the vectors

$$\frac{\partial\Psi}{\partial y_1}, \frac{\partial\Psi}{\partial y_2}, \dots, \frac{\partial\Psi}{\partial y_{n-1}}.$$

The outward unit normal $\nu : \partial\Omega \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$ is orthogonal to this tangent space, and it is given locally by

$$\nu = \frac{\tilde{\nu}}{|\tilde{\nu}|}, \qquad \tilde{\nu} = \frac{\partial \Psi}{\partial y_1} \wedge \frac{\partial \Psi}{\partial y_2} \wedge \cdots \wedge \frac{\partial \Psi}{\partial y_{n-1}},$$

$$\tilde{\nu}_i = \left| \begin{array}{cccc} \partial\Psi_1/\partial y_1 & \partial\Psi_1/\partial\Psi_2 & \dots & \partial\Psi_1/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_{i-1}/\partial y_1 & \partial\Psi_{i-1}/\partial y_2 & \dots & \partial\Psi_{i-1}/\partial y_{n-1} \\ \partial\Psi_{i+1}/\partial y_1 & \partial\Psi_{i+1}/\partial y_2 & \dots & \partial\Psi_{i+1}/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_n/\partial y_1 & \partial\Psi_n/\partial y_2 & \dots & \partial\Psi_n/\partial y_{n-1} \end{array} \right|.$$

EXAMPLE 1.34. For a three-dimensional region with two-dimensional boundary, the outward unit normal is

$$\nu = \frac{(\partial\Psi/\partial y_1) \times (\partial\Psi/\partial y_2)}{|(\partial\Psi/\partial y_1) \times (\partial\Psi/\partial y_2)|}.$$

The restriction of the Euclidean metric on $\mathbb{R}^n$ to the tangent space of the boundary gives a Riemannian metric on the boundary whose volume form defines the surface measure dS. Explicitly, the pull-back of the Euclidean metric

$$\sum_{i=1}^{n} dx_i^2$$

to the boundary under the mapping $x = \Psi(y)$ is the metric

$$\sum_{i=1}^{n} \sum_{p,q=1}^{n-1} \frac{\partial \Psi_i}{\partial y_p} \frac{\partial \Psi_i}{\partial y_q} \, dy_p dy_q.$$

The volume form associated with a Riemannian metric $\sum h_{pq} \, dy_p dy_q$ is

$$\sqrt{\det h} \, dy_1 dy_2 \dots dy_{n-1}.$$

Thus the surface measure on $\partial\Omega$ is given locally by

$$dS = \sqrt{\det\left(D\Psi^t D\Psi\right)} \, dy_1 dy_2 \dots dy_{n-1}$$

where $D\Psi$ is the derivative of the parametrization,

$$D\Psi = \begin{pmatrix} \partial\Psi_1/\partial y_1 & \partial\Psi_1/\partial y_2 & \dots & \partial\Psi_1/\partial y_{n-1} \\ \partial\Psi_2/\partial y_1 & \partial\Psi_2/\partial y_2 & \dots & \partial\Psi_2/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_n/\partial y_1 & \partial\Psi_n/\partial y_2 & \dots & \partial\Psi_n/\partial y_{n-1} \end{pmatrix}.$$

These local expressions may be combined to give a global definition of the surface integral by means of a partition of unity.

EXAMPLE 1.35. In the case of a two-dimensional surface with metric

$$ds^2 = E \, dy_1^2 + 2F \, dy_1 dy_2 + G \, dy_2^2,$$

the element of surface area is

$$dS = \sqrt{EG - F^2} \, dy_1 dy_2.$$

EXAMPLE 1.36. The two-dimensional sphere

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

is a C^∞ submanifold of $\mathbb{R}^3$. A local C^∞-parametrization of

$$U = \mathbb{S}^2 \setminus \left\{(x, 0, z) \in \mathbb{R}^3 : x \geq 0\right\}$$

is given by $\Psi : W \subset \mathbb{R}^2 \to U \subset \mathbb{S}^2$ where

$$\begin{aligned} \Psi(\theta, \phi) &= (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \\ W &= \left\{(\theta, \phi) \in \mathbb{R}^3 : 0 < \theta < 2\pi,\ 0 < \phi < \pi\right\}. \end{aligned}$$

The metric on the sphere is

$$\Psi^* \left(dx^2 + dy^2 + dz^2\right) = \sin^2\phi \, d\theta^2 + d\phi^2$$

and the corresponding surface area measure is

$$dS = \sin\phi \, d\theta d\phi.$$

The integral of a continuous function $f(x, y, z)$ over the sphere that is supported in U is then given by

$$\int_{\mathbb{S}^2} f \, dS = \int_W f\left(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi\right) \sin\phi \, d\theta d\phi.$$

We may use similar rotated charts to cover the points with $x \geq 0$ and $y = 0$.

1.10.3. Representation of a boundary as a graph. An alternative, and computationally simpler, way to represent the boundary of a smooth open set is as a graph. After rotating coordinates, if necessary, we may assume that the nth component of the normal vector to the boundary is nonzero. If $k \geq 1$, the implicit function theorem implies that we may represent a C^k-boundary as a graph

$$x_n = h\left(x_1, x_2, \ldots, x_{n-1}\right)$$

where $h : W \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is in $C^k(W)$ and Ω is given locally by $x_n < h(x_1, \ldots, x_{n-1})$. If the boundary is only Lipschitz, then the implicit function theorem does not apply, and it is not always possible to represent a Lipschitz boundary locally as the region lying above the graph of a Lipschitz continuous function.

If $\partial\Omega$ is C^1, then the outward normal ν is given in terms of h by

$$\nu = \frac{1}{\sqrt{1 + |Dh|^2}} \left(-\frac{\partial h}{\partial x_1}, -\frac{\partial h}{\partial x_2}, \ldots, -\frac{\partial h}{\partial x_{n-1}}, 1\right)$$

and the surface area measure on $\partial\Omega$ is given by

$$dS = \sqrt{1 + |Dh|^2} \, dx_1 dx_2 \ldots dx_{n-1}.$$

EXAMPLE 1.37. Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and $\partial\Omega$ the unit sphere. The upper hemisphere

$$H = \{x \in \partial\Omega : x_n > 0\}$$

is the graph of $x_n = h(x')$ where $h : D \to \mathbb{R}$ is given by

$$h(x') = \sqrt{1 - |x'|^2}, \qquad D = \left\{x' \in \mathbb{R}^{n-1} : |x'| < 1\right\}$$

and we write $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. The surface measure on H is

$$dS = \frac{1}{\sqrt{1 - |x'|^2}}\, dx'$$

and the surface integral of a function $f(x)$ over H is given by

$$\int_H f\, dS = \int_D \frac{f\left(x', h(x')\right)}{\sqrt{1 - |x'|^2}}\, dx'.$$

The integral of a function over $\partial\Omega$ may be computed in terms of such integrals by use of a partition of unity subordinate to an atlas of hemispherical charts.

1.11. Change of variables

We state a theorem for a C^1 change of variables in the Lebesgue integral. A special case is the change of variables from Cartesian to polar coordinates. For proofs, see [**7, 17**].

THEOREM 1.38. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\phi : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism of Ω onto its image $\phi(\Omega)$. If $f : \phi(\Omega) \to \mathbb{R}$ is a nonnegative Lebesgue measurable function or an integrable function, then*

$$\int_{\phi(\Omega)} f(y)\, dy = \int_\Omega f \circ \phi(x) \left|\det D\phi(x)\right|\, dx.$$

We define polar coordinates in $\mathbb{R}^n \setminus \{0\}$ by $x = ry$, where $r = |x| > 0$ and $y \in \partial B_1(0)$ is a point on the unit sphere. In these coordinates, Lebesgue measure has the representation

$$dx = r^{n-1} dr dS(y)$$

where $dS(y)$ is the surface area measure on the unit sphere. We have the following result for integration in polar coordinates.

PROPOSITION 1.39. *If $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, then*

$$\begin{aligned}\int f\, dx &= \int_0^\infty \left[\int_{\partial B_1(0)} f\left(x + ry\right)\, dS(y)\right] r^{n-1}\, dr \\ &= \int_{\partial B_1(0)} \left[\int_0^\infty f\left(x + ry\right)\, r^{n-1}\, dr\right] dS(y).\end{aligned}$$

1.12. Divergence theorem

We state the divergence (or Gauss-Green) theorem.

THEOREM 1.40. *Let $X : \overline{\Omega} \to \mathbb{R}^n$ be a $C^1(\overline{\Omega})$-vector field, and $\Omega \subset \mathbb{R}^n$ a bounded open set with C^1-boundary $\partial\Omega$. Then*

$$\int_\Omega \operatorname{div} X\, dx = \int_{\partial\Omega} X \cdot \nu\, dS.$$

To prove the theorem, we prove it for functions that are compactly supported in a half-space, show that it remains valid under a C^1 change of coordinates with the divergence defined in an appropriately invariant way, and then use a partition of unity to add the results together.

In particular, if $u, v \in C^1(\overline{\Omega})$, then an application of the divergence theorem to the vector field $X = (0, 0, \ldots, uv, \ldots, 0)$, with ith component uv, gives the integration by parts formula

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i} v\, dx + \int_{\partial\Omega} uv\nu_i\, dS.$$

The statement in Theorem 1.40 is, perhaps, the natural one from the perspective of smooth differential geometry. The divergence theorem, however, remains valid under weaker assumptions than the ones in Theorem 1.40. For example, it applies to a cube, whose boundary is not C^1, as well as to other sets with piecewise smooth boundaries.

From the perspective of geometric measure theory, a general form of the divergence theorem holds for Lipschitz vector fields (vector fields whose weak derivative belongs to L^∞) and sets of finite perimeter (sets whose characteristic function has bounded variation). The surface integral is taken over a measure-theoretic boundary with respect to $(n-1)$-dimensional Hausdorff measure, and a measure-theoretic normal exists almost everywhere on the boundary with respect to this measure [**6, 19**].

CHAPTER 2

Laplace's equation

> There can be but one option as to the beauty and utility of this analysis by Laplace; but the manner in which it has hitherto been presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.[1]

Laplace's equation is

$$\Delta u = 0$$

where the Laplacian Δ is defined in Cartesian coordinates by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We may also write $\Delta = \operatorname{div} D$. The Laplacian Δ is invariant under translations (it has constant coefficients) and orthogonal transformations of $\mathbb{R}^n$. A solution of Laplace's equation is called a harmonic function.

Laplace's equation is a linear, scalar equation. It is the prototype of an elliptic partial differential equation, and many of its qualitative properties are shared by more general elliptic PDEs. The non-homogeneous version of Laplace's equation

$$-\Delta u = f$$

is called Poisson's equation. It is convenient to include a minus sign here because Δ is a negative definite operator.

The Laplace and Poisson equations, and their generalizations, arise in many different contexts.

- Potential theory e.g. in the Newtonian theory of gravity, electrostatics, heat flow, and potential flows in fluid mechanics.
- Riemannian geometry e.g. the Laplace-Beltrami operator.
- Stochastic processes e.g. the stationary Kolmogorov equation for Brownian motion.
- Complex analysis e.g. the real and imaginary parts of an analytic function of a single complex variable are harmonic.

As with any PDE, we typically want to find solutions of the Laplace or Poisson equation that satisfy additional conditions. For example, if Ω is a bounded domain in $\mathbb{R}^n$, then the classical Dirichlet problem for Poisson's equation is to find a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

[1]Kelvin and Tait, *Treatise on Natural Philosophy*, 1879

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ are given functions. The classical Neumann problem is to find a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Here, 'classical' refers to the requirement that the functions and derivatives appearing in the problem are defined pointwise as continuous functions. Dirichlet boundary conditions specify the function on the boundary, while Neumann conditions specify the normal derivative. Other boundary conditions, such as mixed (or Robin) and oblique-derivative conditions are also of interest. Also, one may impose different types of boundary conditions on different parts of the boundary (e.g. Dirichlet on one part and Neumann on another).

Here, we mostly follow Evans [**5**] (§2.2), Gilbarg and Trudinger [**10**], and Han and Lin [**12**].

2.1. Mean value theorem

Harmonic functions have the following mean-value property which states that the average value (1.2) of the function over a ball or sphere is equal to its value at the center.

THEOREM 2.1. *Suppose that $u \in C^2(\Omega)$ is harmonic in an open set Ω and $B_r(x) \Subset \Omega$. Then*

$$u(x) = \fint_{B_r(x)} u\,dx, \qquad u(x) = \fint_{\partial B_r(x)} u\,dS. \tag{2.3}$$

PROOF. If $u \in C^2(\Omega)$ and $B_r(x) \Subset \Omega$, then the divergence theorem (Theorem 1.40) implies that

$$\begin{aligned} \int_{B_r(x)} \Delta u\,dx &= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\,dS \\ &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x + ry)\,dS(y) \\ &= r^{n-1} \frac{\partial}{\partial r}\left[\int_{\partial B_1(0)} u(x + ry)\,dS(y)\right]. \end{aligned}$$

Dividing this equation by $\alpha_n r^n$, we find that

$$\fint_{B_r(x)} \Delta u\,dx = \frac{n}{r}\frac{\partial}{\partial r}\left[\fint_{\partial B_r(x)} u\,dS\right]. \tag{2.4}$$

It follows that if u is harmonic, then its mean value over a sphere centered at x is independent of r. Since the mean value integral at $r = 0$ is equal to $u(x)$, the mean value property for spheres follows.

The mean value property for the ball follows from the mean value property for spheres by radial integration. □

The mean value property characterizes harmonic functions and has a remarkable number of consequences. For example, harmonic functions are smooth because local averages over a ball vary smoothly as the ball moves. We will prove this result by mollification, which is a basic technique in the analysis of PDEs.

THEOREM 2.2. *Suppose that $u \in C(\Omega)$ has the mean-value property (2.3). Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.*

PROOF. Let $\eta^\epsilon(x) = \tilde{\eta}^\epsilon(|x|)$ be the standard, radially symmetric mollifier (1.5). If $B_\epsilon(x) \Subset \Omega$, then, using Proposition 1.39 together with the facts that the average of u over each sphere centered at x is equal to $u(x)$ and the integral of η^ϵ is one, we get

$$\begin{aligned}
(\eta^\epsilon * u)(x) &= \int_{B_\epsilon(0)} \eta^\epsilon(y) u(x-y)\, dy \\
&= \int_0^\epsilon \left[\int_{\partial B_1(0)} \eta^\epsilon(rz) u(x - rz)\, dS(z)\right] r^{n-1}\, dr \\
&= n\alpha_n \int_0^\epsilon \left[\fint_{\partial B_r(x)} u\, dS\right] \tilde{\eta}^\epsilon(r) r^{n-1}\, dr \\
&= n\alpha_n u(x) \int_0^\epsilon \tilde{\eta}^\epsilon(r) r^{n-1}\, dr \\
&= u(x) \int \eta^\epsilon(y)\, dy \\
&= u(x).
\end{aligned}$$

Thus, u is smooth since $\eta^\epsilon * u$ is smooth.

If u has the mean value property, then (2.4) shows that

$$\int_{B_r(x)} \Delta u\, dx = 0$$

for every ball $B_r(x) \Subset \Omega$. Since Δu is continuous, it follows that $\Delta u = 0$ in Ω. □

Theorems 2.1–2.2 imply that any C^2-harmonic function is C^∞. The assumption that $u \in C^2(\Omega)$ is, if fact, unnecessary: Weyl showed that if a distribution $u \in \mathcal{D}'(\Omega)$ is harmonic in Ω, then $u \in C^\infty(\Omega)$.

Note that these results say nothing about the behavior of u at the boundary of Ω, which can be nasty. The reverse implication of this observation is that the Laplace equation can take rough boundary data and immediately smooth it to an analytic function in the interior.

EXAMPLE 2.3. Consider the meromorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z}.$$

The real and imaginary parts of f

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = -\frac{y}{x^2 + y^2}$$

are harmonic and C^∞ in, for example, the open unit disc

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\right\}$$

but both are unbounded as $(x, y) \to (0, 0) \in \partial\Omega$.

The boundary behavior of harmonic functions can be much worse than in this example. If $\Omega \subset \mathbb{R}^n$ is any open set, then there exists a harmonic function in Ω such that

$$\liminf_{x \to \xi} u(x) = -\infty, \qquad \limsup_{x \to \xi} u(x) = \infty$$

for all $\xi \in \partial\Omega$. One can construct such a function as a sum of harmonic functions, converging uniformly on compact subsets of Ω, whose terms have singularities on a dense subset of points on $\partial\Omega$.

It is interesting to contrast this result with the the corresponding behavior of holomorphic functions of several variables. An open set $\Omega \subset \mathbb{C}^n$ is said to be a domain of holomorphy if there exists a holomorphic function $f : \Omega \to \mathbb{C}$ which cannot be extended to a holomorphic function on a strictly larger open set. Every open set in $\mathbb{C}$ is a domain of holomorphy, but when $n \geq 2$ there are open sets in $\mathbb{C}^n$ that are not domains of holomorphy, meaning that every holomorphic function on those sets can be extended to a holomorphic function on a larger open set.

2.1.1. Subharmonic and superharmonic functions. The mean value property has an extension to functions that are not necessarily harmonic but whose Laplacian does not change sign.

DEFINITION 2.4. Suppose that Ω is an open set. A function $u \in C^2(\Omega)$ is subharmonic if $\Delta u \geq 0$ in Ω and superharmonic if $\Delta u \leq 0$ in Ω.

A function u is superharmonic if and only if $-u$ is subharmonic, and a function is harmonic if and only if it is both subharmonic and superharmonic. A suitable modification of the proof of Theorem 2.1 gives the following mean value inequality.

THEOREM 2.5. *Suppose that Ω is an open set, $B_r(x) \Subset \Omega$, and $u \in C^2(\Omega)$. If u is subharmonic in Ω, then*

$$u(x) \leq \fint_{B_r(x)} u \, dx, \qquad u(x) \leq \fint_{\partial B_r(x)} u \, dS. \tag{2.5}$$

If u is superharmonic in Ω, then

$$u(x) \geq \fint_{B_r(x)} u \, dx, \qquad u(x) \geq \fint_{\partial B_r(x)} u \, dS. \tag{2.6}$$

It follows from these inequalities that the value of a subharmonic (or superharmonic) function at the center of a ball is less (or greater) than or equal to the value of a harmonic function with the same values on the boundary. Thus, the graphs of subharmonic functions lie below the graphs of harmonic functions and the graphs of superharmonic functions lie above, which explains the terminology. The direction of the inequality ($-\Delta u \leq 0$ for subharmonic functions and $-\Delta u \geq 0$ for superharmonic functions) is more natural when the inequality is stated in terms of the positive operator $-\Delta$.

EXAMPLE 2.6. The function $u(x) = |x|^4$ is subharmonic in $\mathbb{R}^n$ since $\Delta u = 4(n+2)|x|^2 \geq 0$. The function is equal to the constant harmonic function $U(x) = 1$ on the sphere $|x| = 1$, and $u(x) \leq U(x)$ when $|x| \leq 1$.

2.2. Derivative estimates and analyticity

An important feature of Laplace equation is that we can estimate the derivatives of a solution in a ball in terms of the solution on a larger ball. This feature is closely connected with the smoothing properties of the Laplace equation.

THEOREM 2.7. *Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any $1 \le i \le n$,*

$$|\partial_i u(x)| \le \frac{n}{r} \max_{\overline{B}_r(x)} |u|.$$

PROOF. Since u is smooth, differentiation of Laplace's equation with respect to x_i shows that $\partial_i u$ is harmonic, so by the mean value property for balls and the divergence theorem

$$\partial_i u = \fint_{B_r(x)} \partial_i u\, dx = \frac{1}{\alpha_n r^n} \int_{\partial B_r(x)} u\nu_i\, dS.$$

Taking the absolute value of this equation and using the estimate

$$\left| \int_{\partial B_r(x)} u\nu_i\, dS \right| \le n\alpha_n r^{n-1} \max_{\overline{B}_r(x)} |u|$$

we get the result. □

One consequence of Theorem 2.7 is that a bounded harmonic function on $\mathbb{R}^n$ is constant; this is an n-dimensional extension of Liouville's theorem for bounded entire functions.

COROLLARY 2.8. *If $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in $\mathbb{R}^n$, then u is constant.*

PROOF. If $|u| \le M$ on $\mathbb{R}^n$, then Theorem 2.7 implies that

$$|\partial_i u(x)| \le \frac{Mn}{r}$$

for any $r > 0$. Taking the limit as $r \to \infty$, we conclude that $Du = 0$, so u is constant. □

Next we extend the estimate in Theorem 2.7 to higher-order derivatives. We use a somewhat tricky argument that gives sharp enough estimates to prove analyticity.

THEOREM 2.9. *Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any multi-index $\alpha \in \mathbb{N}_0^n$ of order $k = |\alpha|$*

$$|\partial^\alpha u(x)| \le \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x)} |u|.$$

PROOF. We prove the result by induction on $|\alpha| = k$. From Theorem 2.7, the result is true when $k = 1$. Suppose that the result is true when $|\alpha| = k$. If $|\alpha| = k+1$, we may write $\partial^\alpha = \partial_i \partial^\beta$ where $1 \le i \le n$ and $|\beta| = k$. For $0 < \theta < 1$, let

$$\rho = (1-\theta)r.$$

Then, since $\partial^\beta u$ is harmonic and $B_\rho(x) \Subset \Omega$, Theorem 2.7 implies that

$$|\partial^\alpha u(x)| \le \frac{n}{\rho} \max_{\overline{B}_\rho(x)} \left|\partial^\beta u\right|.$$

Suppose that $y \in B_\rho(x)$. Then $B_{r-\rho}(y) \subset B_r(x)$, and using the induction hypothesis we get

$$\left|\partial^\beta u(y)\right| \leq \frac{n^k e^{k-1} k!}{(r-\rho)^k} \max_{\overline{B}_{r-\rho}(y)} |u| \leq \frac{n^k e^{k-1} k!}{r^k \theta^k} \max_{\overline{B}_r(x)} |u| .$$

It follows that

$$|\partial^\alpha u(x)| \leq \frac{n^{k+1} e^{k-1} k!}{r^{k+1}\theta^k(1-\theta)} \max_{\overline{B}_r(x)} |u| .$$

Choosing $\theta = k/(k+1)$ and using the inequality

$$\frac{1}{\theta^k(1-\theta)} = \left(1+\frac{1}{k}\right)^k (k+1) \leq e(k+1)$$

we get

$$|\partial^\alpha u(x)| \leq \frac{n^{k+1} e^k (k+1)!}{r^{k+1}} \max_{\overline{B}_r(x)} |u| .$$

The result follows by induction. □

A consequence of this estimate is that the Taylor series of u converges to u near any point. Thus, we have the following result.

THEOREM 2.10. *If $u \in C^2(\Omega)$ is harmonic in an open set Ω then u is real-analytic in Ω.*

PROOF. Suppose that $x \in \Omega$ and choose $r > 0$ such that $B_{2r}(x) \Subset \Omega$. Since $u \in C^\infty(\Omega)$, we may expand it in a Taylor series with remainder of any order $k \in \mathbb{N}$ to get

$$u(x+h) = \sum_{|\alpha| \leq k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h),$$

where we assume that $|h| < r$. From Theorem 1.21, the remainder is given by

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha \tag{2.7}$$

for some $0 < \theta < 1$.

To estimate the remainder, we use Theorem 2.9 to get

$$|\partial^\alpha u(x+\theta h)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x+\theta h)} |u|.$$

Since $|h| < r$, we have $B_r(x+\theta h) \subset B_{2r}(x)$, so for any $0 < \theta < 1$ we have

$$\max_{\overline{B}_r(x+\theta h)} |u| \leq M, \qquad M = \max_{\overline{B}_{2r}(x)} |u|.$$

It follows that

$$|\partial^\alpha u(x+\theta h)| \leq \frac{M n^k e^{k-1} k!}{r^k}. \tag{2.8}$$

Since $|h^\alpha| \leq |h|^k$ when $|\alpha| = k$, we get from (2.7) and (2.8) that

$$|R_k(x,h)| \leq \frac{M n^k e^{k-1} |h|^k k!}{r^k} \left(\sum_{|\alpha|=k} \frac{1}{\alpha!}\right).$$

The multinomial expansion

$$n^k = (1 + 1 + \cdots + 1)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} = \sum_{|\alpha|=k} \frac{k!}{\alpha!}$$

shows that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{n^k}{k!}.$$

Therefore, we have

$$|R_k(x,h)| \leq \frac{M}{e} \left(\frac{n^2 e|h|}{r} \right)^k.$$

Thus $R_k(x,h) \to 0$ as $k \to \infty$ if

$$|h| < \frac{r}{n^2 e},$$

meaning that the Taylor series of u at any $x \in \Omega$ converges to u in a ball of non-zero radius centered at x. $\square$

It follows that, as for analytic functions, the global values of a harmonic function is determined its values in arbitrarily small balls (or by the germ of the function at a single point).

Corollary 2.11. *Suppose that u, v are harmonic in a connected open set $\Omega \subset \mathbb{R}^n$ and $\partial^\alpha u(\bar{x}) = \partial^\alpha v(\bar{x})$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ at some point $\bar{x} \in \Omega$. Then $u = v$ in Ω.*

Proof. Let

$$F = \{x \in \Omega : \partial^\alpha u(x) = \partial^\alpha v(x) \text{ for all } \alpha \in \mathbb{N}_0^n\}.$$

Then $F \neq \emptyset$, since $\bar{x} \in F$, and F is closed in Ω, since

$$F = \bigcap_{\alpha \in \mathbb{N}_0^n} \left[\partial^\alpha (u - v)\right]^{-1}(0)$$

is an intersection of relatively closed sets. Theorem 2.10 implies that if $x \in F$, then the Taylor series of u, v converge to the same value in some ball centered at x. Thus u, v and all of their partial derivatives are equal in this ball, so F is open. Since Ω is connected, it follows that $F = \Omega$. $\square$

A physical explanation of this property is that Laplace's equation describes an equilibrium solution obtained from a time-dependent solution in the limit of infinite time. For example, in heat flow, the equilibrium is attained as the result of thermal diffusion across the entire domain, while an electrostatic field is attained only after all non-equilibrium electric fields propagate away as electromagnetic radiation. In this infinite-time limit, a change in the field near any point influences the field everywhere else, and consequently complete knowledge of the solution in an arbitrarily small region carries information about the solution in the entire domain.

Although, in principle, a harmonic function function is globally determined by its local behavior near any point, the reconstruction of the global behavior is sensitive to small errors in the local behavior.

EXAMPLE 2.12. Let $\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\, y \in \mathbb{R}\right\}$ and consider for $n \in \mathbb{N}$ the function

$$u_n(x,y) = ne^{-nx}\sin ny,$$

which is harmonic. Then

$$\partial_y^k u_n(x,1) = (-1)^k n^{k+1} e^{-n} \sin nx$$

converges uniformly to zero as $n \to \infty$ for any $k \in \mathbb{N}_0$. Thus, u_n and any finite number of its derivatives are arbitrarily close to zero at $x = 1$ when n is sufficiently large. Nevertheless, $u_n(0,y) = n\sin(ny)$ is arbitrarily large at $y = 0$.

2.3. Maximum principle

The maximum principle states that a non-constant harmonic function cannot attain a maximum (or minimum) at an interior point of its domain. This result implies that the values of a harmonic function in a bounded domain are bounded by its maximum and minimum values on the boundary. Such maximum principle estimates have many uses, but they are typically available only for scalar equations, not systems of PDEs.

THEOREM 2.13. *Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is subharmonic and attains a global maximum value in Ω, then u is constant in Ω.*

PROOF. By assumption, u is bounded from above and attains its maximum in Ω. Let

$$M = \max_{\Omega} u,$$

and consider

$$F = u^{-1}\left(\{M\}\right) = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty and relatively closed in Ω since u is continuous. (A subset F is relatively closed in Ω if $F = \tilde{F} \cap \Omega$ where $\tilde{F}$ is closed in $\mathbb{R}^n$.) If $x \in F$ and $B_r(x) \Subset \Omega$, then the mean value inequality (2.5) for subharmonic functions implies that

$$\fint_{B_r(x)} [u(y) - u(x)]\, dy = \fint_{B_r(x)} u(y)\, dy - u(x) \geq 0.$$

Since u attains its maximum at x, we have $u(y) - u(x) \leq 0$ for all $y \in \Omega$, and it follows that $u(y) = u(x)$ in $B_r(x)$. Therefore F is open as well as closed. Since Ω is connected, and F is nonempty, we must have $F = \Omega$, so u is constant in Ω. $\square$

If Ω is not connected, then u is constant in any connected component of Ω that contains an interior point where u attains a maximum value.

EXAMPLE 2.14. The function $u(x) = |x|^2$ is subharmonic in $\mathbb{R}^n$. It attains a global minimum in $\mathbb{R}^n$ at the origin, but it does not attain a global maximum in any open set $\Omega \subset \mathbb{R}^n$. It does, of course, attain a maximum on any bounded closed set $\overline{\Omega}$, but the attainment of a maximum at a boundary point instead of an interior point does not imply that a subharmonic function is constant.

It follows immediately that superharmonic functions satisfy a minimum principle, and harmonic functions satisfy a maximum and minimum principle.

THEOREM 2.15. *Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is harmonic and attains either a global minimum or maximum in Ω, then u is constant.*

PROOF. Any superharmonic function u that attains a minimum in Ω is constant, since $-u$ is subharmonic and attains a maximum. A harmonic function is both subharmonic and superharmonic. $\square$

EXAMPLE 2.16. The function

$$u(x, y) = x^2 - y^2$$

is harmonic in $\mathbb{R}^2$ (it's the real part of the analytic function $f(z) = z^2$). It has a critical point at 0, meaning that $Du(0) = 0$. This critical point is a saddle-point, however, not an extreme value. Note also that

$$\fint_{B_r(0)} u\, dxdy = \frac{1}{2\pi} \int_0^{2\pi} \left(\cos^2\theta - \sin^2\theta\right)\, d\theta = 0$$

as required by the mean value property.

One consequence of this property is that any nonconstant harmonic function is an open mapping, meaning that it maps opens sets to open sets. This is not true of smooth functions such as $x \mapsto |x|^2$ that attain an interior extreme value.

2.3.1. The weak maximum principle. Theorem 2.13 is an example of a strong maximum principle, because it states that a function which attains an interior maximum is a trivial constant function. This result leads to a weak maximum principle for harmonic functions, which states that the function is bounded inside a domain by its values on the boundary. A weak maximum principle does not exclude the possibility that a non-constant function attains an interior maximum (although it implies that an interior maximum value cannot exceed the maximum value of the function on the boundary).

THEOREM 2.17. *Suppose that Ω is a bounded, connected open set in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \qquad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

PROOF. Since u is continuous and $\overline{\Omega}$ is compact, u attains its global maximum and minimum on $\overline{\Omega}$. If u attains a maximum or minimum value at an interior point, then u is constant by Theorem 2.15, otherwise both extreme values are attained on the boundary. In either case, the result follows. $\square$

Let us give a second proof of this theorem that does not depend on the mean value property. Instead, we use an argument based on the non-positivity of the second derivative at an interior maximum. In the proof, we need to account for the possibility of degenerate maxima where the second derivative is zero.

PROOF. For $\epsilon > 0$, let

$$u^\epsilon(x) = u(x) + \epsilon|x|^2.$$

Then $\Delta u^\epsilon = 2n\epsilon > 0$ since u is harmonic. If u^ϵ attained a local maximum at an interior point, then $\Delta u^\epsilon \le 0$ by the second derivative test. Thus u^ϵ has no interior maximum, and it attains its maximum on the boundary. If $|x| \le R$ for all $x \in \Omega$, it follows that

$$\sup_\Omega u \le \sup_\Omega u^\epsilon \le \sup_{\partial\Omega} u^\epsilon \le \sup_{\partial\Omega} u + \epsilon R^2.$$

Letting $\epsilon \to 0^+$, we get that $\sup_\Omega u \le \sup_{\partial\Omega} u$. An application of the same argument to $-u$ gives $\inf_\Omega u \ge \inf_{\partial\Omega} u$, and the result follows. □

Subharmonic functions satisfy a maximum principle, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, while superharmonic functions satisfy a minimum principle $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

The conclusion of Theorem 2.17 may also be stated as

$$\min_{\partial\Omega} u \le u(x) \le \max_{\partial\Omega} u \qquad \text{for all } x \in \Omega.$$

In physical terms, this means for example that the interior of a bounded region which contains no heat sources or sinks cannot be hotter than the maximum temperature on the boundary or colder than the minimum temperature on the boundary.

The maximum principle gives a uniqueness result for the Dirichlet problem for the Poisson equation.

THEOREM 2.18. *Suppose that Ω is a bounded, connected open set in $\mathbb{R}^n$ and $f \in C(\Omega)$, $g \in C(\partial\Omega)$ are given functions. Then there is at most one solution of the Dirichlet problem (2.1) with $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

PROOF. Suppose that $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy (2.1). Let $v = u_1 - u_2$. Then $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω and $v = 0$ on $\partial\Omega$. The maximum principle implies that $v = 0$ in Ω, so $u_1 = u_2$, and a solution is unique. □

This theorem, of course, does not address the question of whether such a solution exists. In general, the stronger the conditions we impose upon a solution, the easier it is to show uniqueness and the harder it is to prove existence. When we come to prove an existence theorem, we will begin by showing the existence of weaker solutions e.g. solutions in $H^1(\Omega)$ instead of $C^2(\Omega)$. We will then show that these solutions are smooth under suitable assumptions on f, g, and Ω.

2.3.2. Hopf's proof of the maximum principle. Next, we give an alternative proof of the strong maximum principle Theorem 2.13 due to E. Hopf.[2] This proof does not use the mean value property and it works for other elliptic PDEs, not just the Laplace equation.

PROOF. As before, let $M = \max_{\overline{\Omega}} u$ and define

$$F = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty by assumption, and it is relatively closed in Ω since u is continuous.

Now suppose, for contradiction, that $F \ne \Omega$. Then

$$G = \Omega \setminus F$$

is nonempty and open, and the boundary $\partial F \cap \Omega = \partial G \cap \Omega$ is nonempty (otherwise F, G are open and Ω is not connected).

Choose $y \in \partial G \cap \Omega$ and let $d = \operatorname{dist}(y, \partial\Omega) > 0$. There exist points in G that are arbitrarily close to y, so we may choose $x \in G$ such that $|x - y| < d/2$. If

[2] There were two Hopf's (at least): Eberhard Hopf (1902–1983) is associated with the Hopf maximum principle (1927), the Hopf bifurcation theorem, the Wiener-Hopf method in integral equations, and the Cole-Hopf transformation for solving Burgers equation; Heinz Hopf (1894–1971) is associated with the Hopf-Rinow theorem in Riemannian geometry, the Hopf fibration in topology, and Hopf algebras.

$r = \mathrm{dist}(x, F)$, it follows that $0 < r < d/2$, so $\overline{B}_r(x) \subset G$. Moreover, there exists at least one point $\bar{x} \in \partial B_r(x) \cap \partial G$ such that $u(\bar{x}) = M$.

We therefore have the following situation: u is subharmonic in an open set G where $u < M$, the ball $B_r(x)$ is contained in G, and $u(\bar{x}) = M$ for some point $\bar{x} \in \partial B_r(x) \cap \partial G$. The Hopf boundary point lemma, proved below, then implies that

$$\partial_\nu u(\bar{x}) > 0,$$

where ∂_ν is the outward unit normal derivative to the sphere $\partial B_r(z)$

However, since $\bar{x}$ is an interior point of Ω and u attains its maximum value M there, we have $Du(\bar{x}) = 0$, so

$$\partial_\nu u(\bar{x}) = Du(\bar{x}) \cdot \nu = 0.$$

This contradiction proves the theorem. □

Before proving the Hopf lemma, we make a definition.

DEFINITION 2.19. An open set Ω satisfies the interior sphere condition at $\bar{x} \in \partial\Omega$ if there is an open ball $B_r(x)$ contained in Ω such that $\bar{x} \in \partial B_r(x)$

The interior sphere condition is satisfied by open sets with a C^2-boundary, but — as the following example illustrates — it need not be satisfied by open sets with a C^1-boundary, and in that case the conclusion of the Hopf lemma may not hold.

EXAMPLE 2.20. Let

$$u = \Re\left(\frac{z}{\log z}\right) = \frac{x \log r - y\theta}{\log^2 r + \theta^2}$$

where $\log z = \log r + i\theta$ with $-\pi/2 < \theta < \pi/2$. Define

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\, u(x, y) < 0\right\}.$$

Then u is harmonic in Ω, since f is analytic there, and $\partial\Omega$ is C^1 near the origin, with unit outward normal $(-1, 0)$ at the origin. The curvature of $\partial\Omega$, however, becomes infinite at the origin, and the interior sphere condition fails. Moreover, the normal derivative $\partial_\nu u(0, 0) = -u_x(0, 0) = 0$ vanishes at the origin, and it is not strictly positive as would be required by the Hopf lemma.

LEMMA 2.21. *Suppose that $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ is subharmonic in an open set Ω and $u(x) < M$ for every $x \in \Omega$. If $u(\bar{x}) = M$ for some $\bar{x} \in \partial\Omega$ and Ω satisfies the interior sphere condition at $\bar{x}$, then $\partial_\nu u(\bar{x}) > 0$, where ∂_ν is the derivative in the outward unit normal direction to a sphere that touches $\partial\Omega$ at $\bar{x}$.*

PROOF. We want to perturb u to $u^\epsilon = u + \epsilon v$ by a function ϵv with strictly negative normal derivative at $\bar{x}$, while preserving the conditions that $u^\epsilon(\bar{x}) = M$, u^ϵ is subharmonic, and $u^\epsilon < M$ near $\bar{x}$. This will imply that the normal derivative of u at $\bar{x}$ is strictly positive.

We first construct a suitable perturbing function v. Given a ball $B_R(x)$, we want $v \in C^2(\mathbb{R}^n)$ to have the following properties:

(1) $v = 0$ on $\partial B_R(x)$;
(2) $v = 1$ on $\partial B_{R/2}(x)$;
(3) $\partial_\nu v < 0$ on $\partial B_R(x)$;
(4) $\Delta v \geq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$.

We consider without loss of generality a ball $B_R(0)$ centered at 0. Thus, we want to construct a subharmonic function in the annular region $R/2 < |x| < R$ which is 1 on the inner boundary and 0 on the outer boundary, with strictly negative outward normal derivative.

The harmonic function that is equal to 1 on $|x| = R/2$ and 0 on $|x| = R$ is given by

$$u(x) = \frac{1}{2^{n-2}-1}\left[\left(\frac{R}{|x|}\right)^{n-2} - 1\right]$$

(We assume that $n \geq 3$ for simplicity.) Note that

$$\partial_\nu u = -\frac{n-2}{2^{n-2}-1}\frac{1}{R} < 0 \qquad \text{on } |x| = R,$$

so we have room to fit a subharmonic function beneath this harmonic function while preserving the negative normal derivative.

Explicitly, we look for a subharmonic function of the form

$$v(x) = c\left[e^{-\alpha|x|^2} - e^{-\alpha R^2}\right]$$

where c, α are suitable positive constants. We have $v(x) = 0$ on $|x| = R$, and choosing

$$c = \frac{1}{e^{-\alpha R^2/4} - e^{-\alpha R^2}},$$

we have $v(R/2) = 1$. Also, $c > 0$ for $\alpha > 0$. The outward normal derivative of v is the radial derivative, so

$$\partial_\nu v(x) = -2c\alpha|x|e^{-\alpha|x|^2} < 0 \qquad \text{on } |x| = R.$$

Finally, using the expression for the Laplacian in polar coordinates, we find that

$$\Delta v(x) = 2c\alpha\left[2\alpha|x|^2 - n\right]e^{-\alpha|x|^2}.$$

Thus, choosing $\alpha \geq 2n/R^2$, we get $\Delta v < 0$ for $R/2 < |x| < R$, and this gives a function v with the required properties.

By the interior sphere condition, there is a ball $B_R(x) \subset \Omega$ with $\bar{x} \in \partial B_R(x)$. Let

$$M' = \max_{\overline{B}_{R/2}(x)} u < M$$

and define $\epsilon = M - M' > 0$. Let

$$w = u + \epsilon v - M.$$

Then $w \leq 0$ on $\partial B_R(x)$ and $\partial B_{R/2}(x)$ and $\Delta w \geq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$. The maximum principle for subharmonic functions implies that $w \leq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$. Since $w(\bar{x}) = 0$, it follows that $\partial_\nu w(\bar{x}) \geq 0$. Therefore

$$\partial_\nu u(\bar{x}) = \partial_\nu w(\bar{x}) - \epsilon\partial_\nu v(\bar{x}) > 0,$$

which proves the result. □

2.4. Harnack's inequality

The maximum principle gives a basic pointwise estimate for solutions of Laplace's equation, and it has a natural physical interpretation. Harnack's inequality is another useful pointwise estimate, although its physical interpretation is less obvious. It states that if a function is nonnegative and harmonic in a domain, then the ratio of the maximum and minimum of the function on a compactly supported subdomain is bounded by a constant that depends only on the domains. This inequality controls, for example, the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function.

THEOREM 2.22. *Suppose that $\Omega' \Subset \Omega$ is a connected open set that is compactly contained an open set Ω. There exists a constant C, depending only on Ω and Ω', such that if $u \in C(\Omega)$ is a non-negative function with the mean value property, then*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u. \tag{2.9}$$

PROOF. First, we establish the inequality for a compactly contained open ball. Suppose that $x \in \Omega$ and $B_{4R}(x) \subset \Omega$, and let u be any non-negative function with the mean value property in Ω. If $y \in B_R(x)$, then,

$$u(y) = \fint_{B_R(y)} u\,dx \leq 2^n \fint_{B_{2R}(x)} u\,dx$$

since $B_R(y) \subset B_{2R}(x)$ and u is non-negative. Similarly, if $z \in B_R(x)$, then

$$u(z) = \fint_{B_{3R}(z)} u\,dx \geq \left(\frac{2}{3}\right)^n \fint_{B_{2R}(x)} u\,dx$$

since $B_{3R}(z) \supset B_{2R}(x)$. It follows that

$$\sup_{B_R(x)} u \leq 3^n \inf_{B_R(x)} u.$$

Suppose that $\Omega' \Subset \Omega$ and $0 < 4R < \operatorname{dist}(\Omega', \partial\Omega)$. Since $\overline{\Omega'}$ is compact, we may cover Ω' by a finite number of open balls of radius R, where the number N of such balls depends only on Ω' and Ω. Moreover, since Ω' is connected, for any $x, y \in \Omega$ there is a sequence of at most N overlapping balls $\{B_1, B_2, \dots, B_k\}$ such that $B_i \cap B_{i+1} \neq \emptyset$ and $x \in B_1$, $y \in B_k$. Applying the above estimate to each ball and combining the results, we obtain that

$$\sup_{\Omega'} u \leq 3^{nN} \inf_{\Omega'} u.$$

□

In particular, it follows from (2.9) that for any $x, y \in \Omega'$, we have

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y).$$

Harnack's inequality has strong consequences. For example, it implies that if $\{u_n\}$ is a decreasing sequence of harmonic functions in Ω and $\{u_n(x)\}$ is bounded for some $x \in \Omega$, then the sequence converges uniformly on compact subsets of Ω to a function that is harmonic in Ω. By contrast, the convergence of an arbitrary sequence of smooth functions at a single point in no way implies its convergence anywhere else, nor does uniform convergence of smooth functions imply that their limit is smooth.

You can compare this situation with what happens for analytic functions in complex analysis. If $\{f_n\}$ is a sequence of analytic functions

$$f_n : \Omega \subset \mathbb{C} \to \mathbb{C}$$

that converges uniformly on compact subsets of Ω to a function f, then f is also analytic in Ω because uniform convergence implies that the Cauchy integral formula continues to hold for f, and differentiation of this formula implies that f is analytic.

2.5. Green's identities

Green's identities provide the main energy estimates for the Laplace and Poisson equations.

THEOREM 2.23. *If Ω is a bounded C^1 open set in $\mathbb{R}^n$ and $u, v \in C^2(\overline{\Omega})$, then*

$$\int_\Omega u\Delta v\, dx = -\int_\Omega Du \cdot Dv\, dx + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\, dS, \tag{2.10}$$

$$\int_\Omega u\Delta v\, dx = \int_\Omega v\Delta u\, dx + \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS. \tag{2.11}$$

PROOF. Integrating the identity

$$\operatorname{div}(uDv) = u\Delta v + Du \cdot Dv$$

over Ω and using the divergence theorem, we get (2.10). Integrating the identity

$$\operatorname{div}(uDv - vDu) = u\Delta v - v\Delta u,$$

we get (2.11). □

Equations (2.10) and (2.11) are Green's first and second identity, respectively. The second Green's identity implies that the Laplacian Δ is a formally self-adjoint differential operator.

Green's first identity provides a proof of the uniqueness of solutions of the Dirichlet problem based on estimates of L^2-norms of derivatives instead of maximum norms. Such integral estimates are called energy estimates, because in many (though not all) cases these integral norms may be interpreted physically as the energy of a solution.

THEOREM 2.24. *Suppose that Ω is a connected, bounded C^1 open set, $f \in C(\overline{\Omega})$, and $g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solution of the Dirichlet problem (2.1), then $u_1 = u_2$; and if $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions of the Neumann problem (2.2), then $u_1 = u_2 + C$ where $C \in \mathbb{R}$ is a constant.*

PROOF. Let $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω and either $w = 0$ or $\partial w/\partial\nu = 0$ on $\partial\Omega$. Setting $u = w$, $v = w$ in (2.10), it follows that the boundary integral and the integral $\int_\Omega w\Delta w\, dx$ vanish, so that

$$\int_\Omega |Dw|^2\, dx = 0.$$

Therefore $Dw = 0$ in Ω, so w is constant. For the Dirichlet problem, $w = 0$ on $\partial\Omega$ so the constant is zero, and both parts of the result follow. □

2.6. Fundamental solution

We define the fundamental solution or free-space Green's function $\Gamma : \mathbb{R}^n \to \mathbb{R}$ (not to be confused with the Gamma function!) of Laplace's equation by

$$\begin{aligned} \Gamma(x) &= \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}} \qquad \text{if } n \geq 3, \\ \Gamma(x) &= -\frac{1}{2\pi} \log|x| \qquad \text{if } n = 2. \end{aligned} \tag{2.12}$$

The corresponding potential for $n = 1$ is

$$\Gamma(x) = -\frac{1}{2}|x|, \tag{2.13}$$

but we will consider only the multi-variable case $n \geq 2$. (Our sign convention for Γ is the same as Evans [**5**], but the opposite of Gilbarg and Trudinger [**10**].)

2.6.1. Properties of the solution. The potential $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is smooth away from the origin. For $x \neq 0$, we compute that

$$\partial_i \Gamma(x) = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}} \frac{x_i}{|x|}, \tag{2.14}$$

and

$$\partial_{ii}\Gamma(x) = \frac{1}{\alpha_n} \frac{x_i^2}{|x|^{n+2}} - \frac{1}{n\alpha_n} \frac{1}{|x|^n}.$$

It follows that

$$\Delta\Gamma = 0 \qquad \text{if } x \neq 0,$$

so Γ is harmonic in any open set that does not contain the origin. The function Γ is homogeneous of degree $-n+2$, its first derivative is homogeneous of degree $-n+1$, and its second derivative is homogeneous of degree n.

From (2.14), we have for $x \neq 0$ that

$$D\Gamma \cdot \frac{x}{|x|} = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}}$$

Thus we get the following surface integral over a sphere centered at the origin with normal $\nu = x/|x|$:

$$-\int_{\partial B_r(0)} D\Gamma \cdot \nu \, dS = 1. \tag{2.15}$$

As follows from the divergence theorem and the fact that Γ is harmonic in $B_R(0) \setminus B_r(0)$, this integral does not depend on r. The surface integral is not zero, however, as it would be for a function that was harmonic everywhere inside $B_r(0)$, including at the origin. The normalization of the flux integral in (2.15) to one accounts for the choice of the multiplicative constant in the definition of Γ.

The function Γ is unbounded as $x \to 0$ with $\Gamma(x) \to \infty$. Nevertheless, Γ and $D\Gamma$ are locally integrable. For example, the local integrability of $\partial_i\Gamma$ in (2.14) follows from the estimate

$$|\partial_i \Gamma(x)| \leq \frac{C_n}{|x|^{n-1}},$$

since $|x|^{-a}$ is locally integrable on $\mathbb{R}^n$ when $a < n$ (see Example 1.12). The second partial derivatives of Γ are not locally integrable, however, since they are of the order $|x|^{-n}$ as $x \to 0$.

2.6.2. Physical interpretation. Suppose, as in electrostatics, that u is the potential due to a charge distribution with smooth density f and $E = -Du$ is the electric field. Since $-\Delta u = f$, the divergence theorem implies that the flux of E through a boundary $\partial\Omega$ is equal to the to charge inside the enclosed volume, since

$$\int_{\partial\Omega} E \cdot \nu \, dS = \int_{\Omega} (-\Delta u) \, dx = \int_{\Omega} f \, dx.$$

Thus, since $\Delta\Gamma = 0$ for $x \neq 0$ and from (2.15) the flux of $-D\Gamma$ through any sphere centered at the origin is equal to one, we may interpret Γ as the potential due to a point charge located at the origin. In the sense of distribution, Γ satisfies the PDE

$$-\Delta\Gamma = \delta$$

where δ is the delta-function supported at the origin. We refer to such a solution as a Green's function of the Laplacian.

In three space dimensions the electric field $E = -D\Gamma$ is given by

$$E = -\frac{1}{4\pi} \frac{1}{|x|^2} \frac{x}{|x|},$$

corresponding to an inverse-square force directed away from the origin. For gravity, which is always attractive, the force has the opposite sign. This explains the connection between the Laplace and Poisson equations and Newton's inverse square law of gravitation.

As $|x| \to \infty$, the potential $\Gamma(x)$ approaches zero if $n \geq 3$, but $\Gamma(x) \to -\infty$ as $|x| \to \infty$ if $n = 2$. Physically, this corresponds to the fact that only a finite amount of energy is required to remove an object from a point source in three or more space dimensions (for example, to remove a rocket from the earth's gravitational field) but an infinite amount of energy is required to remove an object from a line source in two space dimensions.

We will use the point-source potential Γ to construct solutions of Poisson's equation for rather general right hand sides. The physical interpretation of the method is that we can obtain the potential of a general source by representing the source as a continuous distribution of point sources and superposing the corresponding point-source potential as in (2.24) below. This method, of course, depends crucially on the linearity of the equation.

2.7. The Newtonian potential

Consider the equation

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given function, which for simplicity we assume is smooth and compactly supported.

THEOREM 2.25. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and let*

$$u = \Gamma * f$$

where Γ is the fundamental solution (2.12). Then $u \in C^\infty(\mathbb{R}^n)$ and

$$-\Delta u = f. \tag{2.16}$$

PROOF. Since $f \in C_c^\infty(\mathbb{R}^n)$ and $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, Theorem 1.22 implies that $u \in C^\infty(\mathbb{R}^n)$ and

$$\Delta u = \Gamma * (\Delta f) \tag{2.17}$$

Our objective is to transfer the Laplacian across the convolution from f to Γ.

If $x \notin \operatorname{spt} f$, then we may choose a smooth open set Ω that contains $\operatorname{spt} f$ such that $x \notin \Omega$. Then $\Gamma(x-y)$ is a smooth, harmonic function of y in $\overline{\Omega}$ and f, Df are zero on $\partial\Omega$. Green's theorem therefore implies that

$$\Delta u(x) = \int_\Omega \Gamma(x-y)\Delta f(y)\, dy = \int_\Omega \Delta\Gamma(x-y) f(y)\, dy = 0,$$

which shows that $-\Delta u(x) = f(x)$.

If $x \in \operatorname{spt} f$, we must be careful about the non-integrable singularity in $\Delta\Gamma$. We therefore 'cut out' a ball of radius r about the singularity, apply Green's theorem to the resulting smooth integral, and then take the limit as $r \to 0^+$.

Let Ω be an open set that contains the support of f and define

$$\Omega_r(x) = \Omega \setminus B_r(x)\,. \tag{2.18}$$

Since Δf is bounded with compact support and Γ is locally integrable, the Lebesgue dominated convergence theorem implies that

$$\Gamma * (\Delta f)(x) = \lim_{r\to 0^+} \int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\, dy. \tag{2.19}$$

The potential $\Gamma(x-y)$ is a smooth, harmonic function of y in $\overline{\Omega_r(x)}$. Thus Green's identity (2.11) gives

$$\begin{aligned}
&\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\, dy \\
&\qquad = \int_{\partial\Omega} \left[\Gamma(x-y) D_y f(y)\cdot\nu(y) - D_y\Gamma(x-y)\cdot\nu(y) f(y)\right] dS(y) \\
&\qquad - \int_{\partial B_r(x)} \left[\Gamma(x-y) D_y f(y)\cdot\nu(y) - D_y\Gamma(x-y)\cdot\nu(y) f(y)\right] dS(y)
\end{aligned}$$

where we use the radially outward unit normal on the boundary. The boundary terms on $\partial\Omega$ vanish because f and Df are zero there, so

$$\begin{aligned}
\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\, dy = &-\int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\, dS(y) \\
&+ \int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y) f(y)\, dS(y).
\end{aligned} \tag{2.20}$$

Since Df is bounded and $\Gamma(x) = O(|x|^{n-2})$ if $n \geq 3$, we have

$$\int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\, dS(y) = O(r) \qquad \text{as } r \to 0^+.$$

The integral is $O(r\log r)$ if $n = 2$. In either case,

$$\lim_{r\to 0^+} \int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\, dS(y) = 0. \tag{2.21}$$

For the surface integral in (2.20) that involves $D\Gamma$, we write

$$\begin{aligned}\int_{\partial B_r(x)} & D_y\Gamma(x-y)\cdot\nu(y)f(y)\,dS(y)\\ &= \int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\left[f(y)-f(x)\right]\,dS(y)\\ &\quad + f(x)\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,dS(y).\end{aligned}$$

From (2.15),

$$\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,dS(y) = -1;$$

and, since f is smooth,

$$\int_{\partial B_r(x)} D_y\Gamma(x-y)\left[f(y)-f(x)\right]\,dS(y) = O\left(r^{n-1}\cdot\frac{1}{r^{n-1}}\cdot r\right)\to 0$$

as $r\to 0^+$. It follows that

$$\lim_{r\to 0^+}\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)f(y)\,dS(y) = -f(x). \tag{2.22}$$

Taking the limit of (2.20) as $r\to 0^+$ and using (2.21) and (2.22) in the result, we get

$$\lim_{r\to 0^+}\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\,dy = -f(x).$$

The use of this equation in (2.19) shows that

$$\Gamma * (\Delta f) = -f, \tag{2.23}$$

and the use of (2.23) in (2.17) gives (2.16). □

Equation (2.23) is worth noting: it provides a representation of a function $f\in C_c^\infty(\mathbb{R}^n)$ as a convolution of its Laplacian with the Newtonian potential.

The potential u associated with a source distribution f is given by

$$u(x) = \int \Gamma(x-y)f(y)\,dy. \tag{2.24}$$

We call u the Newtonian potential of f. We may interpret $u(x)$ as a continuous superposition of potentials proportional to $\Gamma(x-y)$ due to point sources of strength $f(y)\,dy$ located at y.

If $n\geq 3$, the potential $\Gamma * f(x)$ of a compactly supported, integrable function approaches zero as $|x|\to\infty$. We have

$$\Gamma * f(x) = \frac{1}{n(n-2)\alpha_n|x|^{n-2}}\int\left(\frac{|x|}{|x-y|}\right)^{n-2} f(y)\,dy,$$

and by the Lebesgue dominated convergence theorem,

$$\lim_{|x|\to\infty}\int\left(\frac{|x|}{|x-y|}\right)^{n-2} f(y)\,dy = \int f(y)\,dy.$$

Thus, the asymptotic behavior of the potential is the same as that of a point source whose charge is equal to the total charge of the source density f. If $n=2$, the potential, in general, grows logarithmically as $|x|\to\infty$.

If $n \geq 3$, Liouville's theorem (Corollary 2.8) implies that the Newtonian potential $\Gamma * f$ is the unique solution of $-\Delta u = f$ such that $u(x) \to 0$ as $x \to \infty$. (If u_1, u_2 are solutions, then $v = u_1 - u_2$ is harmonic in $\mathbb{R}^n$ and approaches 0 as $x \to \infty$; thus v is bounded and therefore constant, so $v = 0$.) If $n = 2$, then a similar argument shows that any solution of Poisson's equation such that $Du(x) \to 0$ as $|x| \to \infty$ differs from the Newtonian potential by a constant.

2.7.1. Second derivatives of the potential. In order to study the regularity of the Newtonian potential u in terms of f, we derive an integral representation for its second derivatives.

We write $\partial_i \partial_j = \partial_{ij}$, and let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

denote the Kronecker delta. In the following $\partial_i \Gamma(x - y)$ denotes the ith partial derivative of Γ evaluated at $x - y$, with similar notation for other derivatives. Thus,

$$\frac{\partial}{\partial y_i} \Gamma(x - y) = -\partial_i \Gamma(x - y).$$

Theorem 2.26. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and $u = \Gamma * f$ where Γ is the Newtonian potential (2.12). If Ω is any smooth open set that contains the support of f, then*

$$\begin{aligned} \partial_{ij} u(x) = &\int_\Omega \partial_{ij} \Gamma(x - y) \big[f(y) - f(x) \big] \, dy \\ &- f(x) \int_{\partial \Omega} \partial_i \Gamma(x - y) \nu_j(y) \, dS(y). \end{aligned} \tag{2.25}$$

Proof. As before, the result is straightforward to prove if $x \notin \operatorname{spt} f$. We choose $\Omega \supset \operatorname{spt} f$ such that $x \notin \overline{\Omega}$. Then Γ is smooth on $\overline{\Omega}$ so we may differentiate under the integral sign to get

$$\partial_{ij} u(x) = \int_\Omega \partial_{ij} \Gamma(x - y) f(y) \, dy.,$$

which is (2.25) with $f(x) = 0$.

If $x \in \operatorname{spt} f$, we follow a similar procedure to the one used in the proof of Theorem 2.25: We differentiate under the integral sign in the convolution $u = \Gamma * f$ on f, cut out a ball of radius r about the singularity in Γ, apply Greens' theorem, and let $r \to 0^+$.

In detail, define $\Omega_r(x)$ as in (2.18), where $\Omega \supset \operatorname{spt} f$ is a smooth open set. Since Γ is locally integrable, the Lebesgue dominated convergence theorem implies that

$$\partial_{ij} u(x) = \int_\Omega \Gamma(x - y) \partial_{ij} f(y) \, dy = \lim_{r \to 0^+} \int_{\Omega_r(x)} \Gamma(x - y) \partial_{ij} f(y) \, dy. \tag{2.26}$$

For $x \neq y$, we have the identity

$$\begin{aligned} &\Gamma(x - y) \partial_{ij} f(y) - \partial_{ij} \Gamma(x - y) f(y) \\ &\qquad = \frac{\partial}{\partial y_i} \left[\Gamma(x - y) \partial_j f(y) \right] + \frac{\partial}{\partial y_j} \left[\partial_i \Gamma(x - y) f(y) \right]. \end{aligned}$$

Thus, using Green's theorem, we get

$$\begin{aligned}\int_{\Omega_r(x)} \Gamma(x-y)\partial_{ij} f(y)\,dy &= \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y) f(y)\,dy \\ &\quad - \int_{\partial B_r(x)} \left[\Gamma(x-y)\partial_j f(y)\nu_i(y) + \partial_i\Gamma(x-y) f(y)\nu_j(y)\right]\,dS(y).\end{aligned} \tag{2.27}$$

In (2.27), ν denotes the radially outward unit normal vector on $\partial B_r(x)$, which accounts for the minus sign of the surface integral; the integral over the boundary $\partial\Omega$ vanishes because f is identically zero there.

We cannot take the limit of the integral over $\Omega_r(x)$ directly, since $\partial_{ij}\Gamma$ is not locally integrable. To obtain a limiting integral that is convergent, we write

$$\begin{aligned}&\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y) f(y)\,dy \\ &\qquad = \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy + f(x)\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\,dy \\ &\qquad = \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy \\ &\qquad\quad - f(x)\left[\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) - \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\right].\end{aligned}$$

Using this expression in (2.27) and using the result in (2.26), we get

$$\begin{aligned}\partial_{ij}u(x) = \lim_{r\to 0^+} &\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy \\ &- f(x)\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) \\ &- \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\big[f(y)-f(x)\big]\nu_j(y)\,dS(y) \\ &- \int_{\partial B_r(x)} \Gamma(x-y)\partial_j f(y)\nu_i(y)\,dS(y).\end{aligned} \tag{2.28}$$

Since f is smooth, the function $y \mapsto \partial_{ij}\Gamma(x-y)\left[f(y)-f(x)\right]$ is integrable on Ω, and by the Lebesgue dominated convergence theorem

$$\lim_{r\to 0^+}\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy = \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy.$$

We also have

$$\begin{aligned}&\lim_{r\to 0^+}\int_{\partial B_r(x)} \partial_i\Gamma(x-y)\big[f(y)-f(x)\big]\nu_j(y)\,dS(y) = 0, \\ &\lim_{r\to 0^+}\int_{\partial B_r(x)} \Gamma(x-y)\partial_j f(y)\nu_i(y)\,dS(y) = 0.\end{aligned}$$

Using these limits in (2.28), we get (2.25). □

Note that if $\Omega' \supset \Omega \supset \operatorname{spt} f$, then writing

$$\Omega' = \Omega \cup (\Omega' \setminus \Omega)$$

and using the divergence theorem, we get

$$\begin{aligned}
&\int_{\Omega'} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy - f(x)\int_{\partial\Omega'} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\\
&\quad= \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy\\
&\qquad - f(x)\left[\int_{\partial\Omega'} \partial_i\Gamma(x-y)\nu_j(x-y)\,dS(y) + \int_{\Omega'\setminus\Omega} \partial_{ij}\Gamma(x-y)\,dy\right]\\
&\quad= \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy - f(x)\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y).
\end{aligned}$$

Thus, the expression on the right-hand side of (2.25) does not depend on Ω provided that it contains the support of f. In particular, we can choose Ω to be a sufficiently large ball centered at x.

Corollary 2.27. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and $u = \Gamma * f$ where Γ is the Newtonian potential (2.12). Then*

$$\partial_{ij}u(x) = \int_{B_R(x)} \partial_{ij}\Gamma(x-y)\,[f(y)-f(x)]\,dy - \frac{1}{n}f(x)\delta_{ij} \tag{2.29}$$

where $B_R(x)$ is any open ball centered at x that contains the support of f.

Proof. In (2.25), we choose $\Omega = B_R(x) \supset \operatorname{spt} f$. From (2.14), we have

$$\begin{aligned}
\int_{\partial B_R(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) &= \int_{\partial B_R(x)} \frac{-(x_i-y_i)}{n\alpha_n|x-y|^n}\,\frac{y_j-x_j}{|y-x|}\,dS(y)\\
&= \int_{\partial B_R(0)} \frac{y_iy_j}{n\alpha_n|y|^{n+1}}\,dS(y)
\end{aligned}$$

If $i \ne j$, then y_iy_j is odd under a reflection $y_i \mapsto -y_i$, so this integral is zero. If $i = j$, then the value of the integral does not depend on i, since we may transform the i-integral into an i'-integral by a rotation. Therefore

$$\begin{aligned}
\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{y_i^2}{|y|^{n+1}}\,dS(y) &= \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{y_i^2}{|y|^{n+1}}\,dS(y)\right)\\
&= \frac{1}{n}\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{1}{|y|^{n-1}}\,dS(y)\\
&= \frac{1}{n}.
\end{aligned}$$

It follows that

$$\int_{\partial B_R(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) = \frac{1}{n}\delta_{ij}.$$

Using this result in (2.25), we get (2.29). □

2.7.2. Hölder estimates. We want to derive estimates of the derivatives of the Newtonian potential $u = \Gamma * f$ in terms of the source density f. We continue to assume that $f \in C_c^\infty(\mathbb{R}^n)$; the estimates extend by a density argument to any Hölder-continuous function f with compact support (or sufficiently rapid decay at infinity).

In one space dimension, a solution of the ODE

$$-u'' = f$$

is given in terms of the potential (2.13) by

$$u(x) = -\frac{1}{2}\int |x-y|\, f(y)\, dy.$$

If $f \in C_c(\mathbb{R})$, then obviously $u \in C^2(\mathbb{R})$ and $\max|u''| = \max|f|$.

In more than one space dimension, however, it is not possible estimate the maximum norm of the second derivative D^2u of the potential $u = \Gamma * f$ in terms of the maximum norm of f, and there exist functions $f \in C_c(\mathbb{R}^n)$ for which $u \notin C^2(\mathbb{R}^n)$.

Nevertheless, if we measure derivatives in an appropriate way, we gain two derivatives in solving the Laplace equation (and other second-order elliptic PDEs). The fact that in inverting the Laplacian we gain as many derivatives as the order of the PDE is the essential point of elliptic regularity theory; this does not happen for many other types of PDEs, such as hyperbolic PDEs.

In particular, if we measure derivatives in terms of their Hölder continuity, we can estimate the $C^{2,\alpha}$-norm of u in terms of the $C^{0,\alpha}$-norm of f. These Hölder estimates were used by Schauder[3] to develop a general existence theory for elliptic PDEs with Hölder continuous coefficients, typically referred to as the Schauder theory [**10**].

Here, we will derive Hölder estimates for the Newtonian potential.

THEOREM 2.28. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$ and $0 < \alpha < 1$. If $u = \Gamma * f$ where Γ is the Newtonian potential (2.12), then*

$$[\partial_{ij}u]_{0,\alpha} \le C\, [f]_{0,\alpha}$$

where $[\cdot]_{0,\alpha}$ denotes the Hölder semi-norm (1.1) and C is a constant that depends only on α and n.

PROOF. Let Ω be a smooth open set that contains the support of f. We write (2.25) as

$$\partial_{ij}u = Tf - fg \tag{2.30}$$

where the linear operator

$$T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is defined by

$$Tf(x) = \int_\Omega K(x-y)\,[f(y) - f(x)]\, dy, \qquad K = \partial_{ij}\Gamma,$$

and the function $g : \mathbb{R}^n \to \mathbb{R}$ is given by

$$g(x) = \int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\, dS(y). \tag{2.31}$$

If $x, x' \in \mathbb{R}^n$, then

$$\partial_{ij}u(x) - \partial_{ij}u(x') = Tf(x) - Tf(x') - [f(x)g(x) - f(x')g(x')]$$

[3]Juliusz Schauder (1899–1943) was a Polish mathematician. In addition to the Schauder theory for elliptic PDEs, he is known for the Leray-Schauder fixed point theorem, and Schauder bases of a Banach space. He was killed by the Nazi's while they occupied Lvov during the second world war.

The main part of the proof is to estimate the difference of the terms that involve Tf.

In order to do this, let

$$\bar{x} = \frac{1}{2}(x + x'), \qquad \delta = |x - x'|,$$

and choose Ω so that it contains $B_{2\delta}(\bar{x})$. We have

$$\begin{aligned} &Tf(x) - Tf(x') \\ &\quad = \int_\Omega \{K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]\}\, dy. \end{aligned} \tag{2.32}$$

We will separate the the integral over Ω in (2.32) into two parts: (a) $|y - \bar{x}| < \delta$; (b) $|y - \bar{x}| \geq \delta$. In region (a), which contains the points $y = x$, $y = x'$ where K is singular, we will use the Hölder continuity of f and the smallness of the integration region to estimate the integral. In region (b), we will use the Hölder continuity of f and the smoothness of K to estimate the integral.

(a) Suppose that $|y - \bar{x}| < \delta$, meaning that $y \in B_\delta(\bar{x})$. Then

$$|x - y| \leq |x - \bar{x}| + |\bar{x} - y| \leq \frac{3}{2}\delta,$$

so $y \in B_{3\delta/2}(x)$, and similarly for x'. Using the Hölder continuity of f and the fact that K is homogeneous of degree $-n$, we have

$$\begin{aligned} &|K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]| \\ &\qquad\qquad \leq C[f]_{0,\alpha}\left\{|x-y|^{\alpha-n} + |x'-y|^{\alpha-n}\right\}. \end{aligned}$$

Thus, using C to denote a generic constant depending on α and n, we get

$$\begin{aligned} &\int_{B_\delta(\bar{x})} |K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]|\, dy \\ &\qquad \leq C[f]_{0,\alpha}\int_{B_\delta(\bar{x})}\left[|x-y|^{\alpha-n} + |x'-y|^{\alpha-n}\right] dy \\ &\qquad \leq C[f]_{0,\alpha}\int_{B_{3\delta/2}(0)} |y|^{\alpha-n} dy \\ &\qquad \leq C[f]_{0,\alpha}\,\delta^\alpha. \end{aligned}$$

(b) Suppose that $|y - \bar{x}| \geq \delta$. We write

$$\begin{aligned} &K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')] \\ &\quad = [K(x-y) - K(x'-y)][f(y) - f(x)] - K(x'-y)[f(x) - f(x')] \end{aligned} \tag{2.33}$$

and estimate the two terms on the right hand side separately. For the first term, we use the the Hölder continuity of f and the smoothness of K; for the second term we use the Hölder continuity of f and the divergence theorem to estimate the integral of K.

(b1) Since DK is homogeneous of degree $-(n+1)$, the mean value theorem implies that

$$|K(x-y) - K(x'-y)| \leq C\frac{|x - x'|}{|\xi - y|^{n+1}}$$

for $\xi = \theta x + (1-\theta)x'$ with $0 < \theta < 1$. Using this estimate and the Hölder continuity of f, we get

$$|[K(x-y) - K(x'-y)]\,[f(y) - f(x)]| \leq C\,[f]_{0,\alpha}\,\delta \frac{|y-x|^\alpha}{|\xi - y|^{n+1}}.$$

We have

$$\begin{aligned}
|y-x| &\leq |y-\bar{x}| + |\bar{x}-x| = |y-\bar{x}| + \frac{1}{2}\delta \leq \frac{3}{2}|y-\bar{x}|,\\
|\xi - y| &\geq |y-\bar{x}| - |\bar{x}-\xi| \geq |y-\bar{x}| - \frac{1}{2}\delta \geq \frac{1}{2}|y-\bar{x}|.
\end{aligned}$$

It follows that

$$|[K(x-y) - K(x'-y)]\,[f(y) - f(x)]| \leq C\,[f]_{0,\alpha}\,\delta |y-\bar{x}|^{\alpha-n-1}.$$

Thus,

$$\begin{aligned}
&\int_{\Omega\setminus B_\delta(\bar{x})} |[K(x-y) - K(x'-y)]\,[f(y) - f(x)]|\;dy\\
&\qquad \leq \int_{\mathbb{R}^n\setminus B_\delta(\bar{x})} |[K(x-y) - K(x'-y)]\,[f(y) - f(x)]|\;dy\\
&\qquad \leq C\,[f]_{0,\alpha}\,\delta \int_{|y|\geq\delta} |y|^{\alpha-n-1}\,dy\\
&\qquad \leq C\,[f]_{0,\alpha}\,\delta^\alpha.
\end{aligned}$$

Note that the integral does not converge at infinity if $\alpha = 1$; this is where we require $\alpha < 1$.

(b2) To estimate the second term in (2.33), we suppose that $\Omega = B_R(\bar{x})$ where $B_R(\bar{x})$ contains the support of f and $R \geq 2\delta$. (All of the estimates above apply for this choice of Ω.) Writing $K = \partial_{ij}\Gamma$ and using the divergence theorem we get

$$\begin{aligned}
&\int_{B_R(\bar{x})\setminus B_\delta(\bar{x})} K(x-y)\,dy\\
&\qquad = \int_{\partial B_R(\bar{x})} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) - \int_{\partial B_\delta(\bar{x})} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y).
\end{aligned}$$

If $y \in \partial B_R(\bar{x})$, then

$$|x-y| \geq |y-\bar{x}| - |\bar{x}-x| \geq R - \frac{1}{2}\delta \geq \frac{3}{4}R;$$

and If $y \in \partial B_\delta(\bar{x})$, then

$$|x-y| \geq |y-\bar{x}| - |\bar{x}-x| \geq \delta - \frac{1}{2}\delta \geq \frac{1}{2}\delta.$$

Thus, using the fact that $D\Gamma$ is homogeneous of degree $-n+1$, we compute that

$$\int_{\partial B_R(\bar{x})} |\partial_i\Gamma(x-y)\nu_j(y)|\;dS(y) \leq CR^{n-1}\frac{1}{R^{n-1}} \leq C \tag{2.34}$$

and

$$\int_{\partial B_\delta(\bar{x})} |\partial_i\Gamma(x-y)\nu_j(y)\,dS(y)|\,C\delta^{n-1}\frac{1}{\delta^{n-1}} \leq C$$

Thus, using the Hölder continuity of f, we get

$$\left|[f(x)-f(x')]\int_{\Omega\setminus B_\delta(\bar{x})} K(x'-y)\,dy\right| \le C\,[f]_{0,\alpha}\,\delta^\alpha.$$

Putting these estimates together, we conclude that

$$|Tf(x)-Tf(x')| \le C\,[f]_{0,\alpha}\,|x-x'|^\alpha$$

where C is a constant that depends only on α and n.

(c) Finally, to estimate the Hölder norm of the remaining term fg in (2.30), we continue to assume that $\Omega = B_R(\bar{x})$. From (2.31),

$$g(\bar{x}+h) = \int_{\partial B_R(0)} \partial_i\Gamma(h-y)\nu_j(y)\,dS(y).$$

Changing $y \mapsto -y$ in the integral, we find that $g(\bar{x}+h) = g(\bar{x}-h)$. Hence $g(x) = g(x')$. Moreover, from (2.34), we have $|g(x)| \le C$. It therefore follows that

$$|f(x)g(x)-f(x')g(x')| \le C\,|f(x)-f(x')| \le C\,[f]_{0,\alpha}\,|x-x'|^\alpha,$$

which completes the proof. □

These Hölder estimates, and their generalizations, are fundamental to theory of elliptic PDEs. Their derivation by direct estimation of the Newtonian potential is only one of many methods to obtain them (although it was the original method). For example, they can also be obtained by the use of Campanato spaces, which provide Hölder estimates in terms of suitable integral norms [**12**], or by the use of Littlewood-Payley theory, which provides Hölder estimates in terms of dyadic decompositions of the Fourier transform [**2**].

2.8. Singular integral operators

Using (2.29), we may define a linear operator

$$T_{ij} : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

that gives the second derivatives of a function in terms of its Laplacian,

$$\partial_{ij}u = T_{ij}\Delta u.$$

Explicitly,

$$T_{ij}f(x) = \int_{B_R(x)} K_{ij}(x-y)\,[f(y)-f(x)]\,dy + \frac{1}{n}f(x)\delta_{ij} \tag{2.35}$$

where $B_R(x) \supset \operatorname{spt} f$ and $K_{ij} = -\partial_{ij}\Gamma$ is given by

$$K_{ij}(x) = \frac{1}{\alpha_n|x|^n}\left(\frac{1}{n}\delta_{ij} - \frac{x_ix_j}{|x|^2}\right). \tag{2.36}$$

This function is homogeneous of degree $-n$, the borderline power for integrability, so it is not locally integrable. Thus, Young's inequality does not imply that convolution with K_{ij} is a bounded operator on L^∞_{loc}, which explains why we cannot bound the maximum norm of D^2u in terms of the maximum norm of f.

The kernel K_{ij} in (2.36) has zero integral over any sphere, meaning that

$$\int_{B_R(0)} K_{ij}(y)\,dS(y) = 0.$$

Thus, we may alternatively write T_{ij} as

$$\begin{aligned}
T_{ij}f(x) - \frac{1}{n}f(x)\delta_{ij} &= \lim_{\epsilon\to 0^+}\int_{B_R(x)\setminus B_\epsilon(x)} K_{ij}(x-y)\left[f(y)-f(x)\right]\,dy \\
&= \lim_{\epsilon\to 0^+}\int_{B_R(x)\setminus B_\epsilon(x)} K_{ij}(x-y)f(y)\,dy \\
&= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} K_{ij}(x-y)f(y)\,dy.
\end{aligned}$$

This is an example of a singular integral operator.

The operator T_{ij} can also be expressed in terms of the Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n}\int f(x)e^{-i\cdot\xi}\,dx$$

as

$$\widehat{(T_{ij}f)}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}\hat{f}(\xi).$$

Since the multiplier $m_{ij}:\mathbb{R}^n\to\mathbb{R}$ defined by

$$m_{ij}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}$$

belongs to $L^\infty(\mathbb{R}^n)$, it follows from Plancherel's theorem that T_{ij} extends to a bounded linear operator on $L^2(\mathbb{R}^n)$.

In more generality, consider a function $K:\mathbb{R}^n\to\mathbb{R}$ that is continuously differentiable in $\mathbb{R}^n\setminus 0$ and satisfies the following conditions:

$$\begin{aligned}
&K(\lambda x) = \frac{1}{\lambda^n}K(x) \qquad \text{for } \lambda>0; \\
&\int_{\partial B_R(0)} K\,dS = 0 \qquad \text{for } R>0.
\end{aligned} \tag{2.37}$$

That is, K is homogeneous of degree $-n$, and its integral over any sphere centered at zero is zero. We may then write

$$K(x) = \frac{\Omega\left(\hat{x}\right)}{|x|^n}, \qquad \hat{x} = \frac{x}{|x|}$$

where $\Omega:\mathbb{S}^{n-1}\to\mathbb{R}$ is a C^1-function such that

$$\int_{\mathbb{S}^{n-1}}\Omega\,dS = 0.$$

We define a singular integral operator $T:C_c^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$ of convolution type with smooth, homogeneous kernel K by

$$Tf(x) = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} K(x-y)f(y)\,dy. \tag{2.38}$$

This operator is well-defined, since if $B_R(x) \supset \operatorname{spt} f$, we may write

$$\begin{aligned} Tf(x) &= \lim_{\epsilon \to 0^+} \int_{B_R(x) \setminus B_\epsilon(x)} K(x-y) f(y)\, dy. \\ &= \lim_{\epsilon \to 0^+} \left\{ \int_{B_R(x) \setminus B_\epsilon(x)} K(x-y) \left[f(y) - f(x) \right] \, dy \right. \\ &\qquad\qquad \left. + f(x) \int_{B_R(x) \setminus B_\epsilon(x)} K(x-y)\, dy \right\} \\ &= \int_{B_R(x)} K(x-y) \left[f(y) - f(x) \right] \, dy. \end{aligned}$$

Here, we use the dominated convergence theorem and the fact that

$$\int_{B_R(0) \setminus B_\epsilon(0)} K(y)\, dy = 0$$

since K has zero mean over spheres centered at the origin. Thus, the cancelation due to the fact that K has zero mean over spheres compensates for the non-integrability of K at the origin to give a finite limit.

Calderón and Zygmund (1952) proved that such operators, and generalizations of them, extend to bounded linear operators on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$ (see e.g. [**3**]). As a result, we also 'gain' two derivatives in inverting the Laplacian when derivatives are measured in L^p for $1 < p < \infty$.

CHAPTER 3

Sobolev spaces

We will give only the most basic results here. For more information, see Shkoller [**16**], Evans [**5**] (Chapter 5), and Leoni [**14**]. A standard reference is [**1**].

3.1. Weak derivatives

Suppose, as usual, that Ω is an open set in $\mathbb{R}^n$.

DEFINITION 3.1. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f\partial_i\phi\, dx = -\int_\Omega g_i\phi\, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

The function g_i is called the weak ith partial derivative of f, and is denoted by $\partial_i f$.

Thus, for weak derivatives, the integration by parts formula

$$\int_\Omega f\partial_i\phi\, dx = -\int_\Omega \partial_i f\phi\, dx$$

holds by definition for all $\phi \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^1_{\mathrm{loc}}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence. Moreover, the weak derivative of a continuously differentiable function agrees with the pointwise derivative. The existence of a weak derivative is, however, *not* equivalent to the existence of a pointwise derivative almost everywhere; see Examples 3.4 and 3.5.

Unless stated otherwise, we will always interpret derivatives as weak derivatives, and we use the same notation for weak derivatives and continuous pointwise derivatives.

Higher-order weak derivatives are defined in a similar way.

DEFINITION 3.2. Suppose that $\alpha \in \mathbb{N}_0^n$ is a multi-index. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ if

$$\int_\Omega (\partial^\alpha f)\, \phi\, dx = (-1)^{|\alpha|} \int_\Omega f\, (\partial^\alpha \phi)\; dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

3.2. Examples

Let us consider some examples of weak derivatives that illustrate the definition. We denote the weak derivative of a function of a single variable by a prime.

EXAMPLE 3.3. Define $f \in C(\mathbb{R})$ by

$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

We also write $f(x) = x_+$. Then f is weakly differentiable, with

$$f' = \chi_{[0,\infty)}, \tag{3.1}$$

where $\chi_{[0,\infty)}$ is the step function

$$\chi_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The choice of the value of $f'(x)$ at $x = 0$ is irrelevant, since the weak derivative is only defined up to pointwise almost everwhere equivalence. To prove (3.1), note that for any $\phi \in C_c^\infty(\mathbb{R})$, an integration by parts gives

$$\int f\phi'\,dx = \int_0^\infty x\phi'\,dx = -\int_0^\infty \phi\,dx = -\int \chi_{[0,\infty)}\phi\,dx.$$

EXAMPLE 3.4. The discontinuous function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

is not weakly differentiable. To prove this, note that for any $\phi \in C_c^\infty(\mathbb{R})$,

$$\int f\phi'\,dx = \int_0^\infty \phi'\,dx = -\phi(0).$$

Thus, the weak derivative $g = f'$ would have to satisfy

$$\int g\phi\,dx = \phi(0) \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}). \tag{3.2}$$

Assume for contradiction that $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfies (3.2). By considering test functions with $\phi(0) = 0$, we see that g is equal to zero pointwise almost everywhere, and then (3.2) does not hold for test functions with $\phi(0) \neq 0$.

The pointwise derivative of the discontinuous function f in the previous example exists and is zero except at 0, where the function is discontinuous, but the function is not weakly differentiable. The next example shows that even a continuous function that is pointwise differentiable almost everywhere need not have a weak derivative.

EXAMPLE 3.5. Let $f \in C(\mathbb{R})$ be the Cantor function, which may be constructed as a uniform limit of piecewise constant functions defined on the standard 'middle-thirds' Cantor set C. For example, $f(x) = 1/2$ for $1/3 \leq x \leq 2/3$, $f(x) = 1/4$ for $1/9 \leq x \leq 2/9$, $f(x) = 3/4$ for $7/9 \leq x \leq 8/9$, and so on.[1] Then f is not weakly differentiable. To see this, suppose that $f' = g$ where

$$\int g\phi\,dx = -\int f\phi'\,dx$$

[1]The Cantor function is given explicitly by: $f(x) = 0$ if $x \leq 0$; $f(x) = 1$ if $x \geq 1$;

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{c_n}{2^n}$$

if $x = \sum_{n=1}^{\infty} c_n/3^n$ with $c_n \in \{0, 2\}$ for all $n \in \mathbb{N}$; and

$$f(x) = \frac{1}{2}\sum_{n=1}^{N} \frac{c_n}{2^n} + \frac{1}{2^{N+1}}$$

if $x = \sum_{n=1}^{\infty} c_n/3^n$, with $c_n \in \{0, 2\}$ for $1 \leq n < k$ and $c_k = 1$.

for all test functions ϕ. The complement of the Cantor set in $[0,1]$ is a union of open intervals,

$$[0,1]\setminus C = \left(\frac{1}{3},\frac{2}{3}\right) \cup \left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right) \cup \dots,$$

whose measure is equal to one. Taking test functions ϕ whose supports are compactly contained in one of these intervals, call it I, and using the fact that $f = c_I$ is constant on I, we find that

$$\int g\phi\,dx = -\int_I f\phi'\,dx = -c_I \int_I \phi'\,dx = 0.$$

It follows that $g = 0$ pointwise a.e. on $[0,1]\setminus C$, and hence pointwise a.e. on $[0,1]$ since C has measure zero. Thus,

$$\int f\phi'\,dx = 0 \qquad \text{for all } \phi \in C_c^\infty(0,1). \tag{3.3}$$

We claim that (3.3) implies that f is equivalent to a constant function in $(0,1)$, which is a contradiction.

To prove the last claim, choose a fixed test function $\eta \in C_c^\infty(0,1)$ whose integral is equal to one. We may represent an arbitrary test function $\phi \in C_c^\infty(0,1)$ as

$$\phi = A\eta + \psi'$$

where $A \in \mathbb{R}$ and $\psi \in C_c^\infty(0,1)$ are given by

$$A = \int_0^1 \phi\,dx, \qquad \psi(x) = \int_0^x [\phi(t) - A\eta(t)]\,dt.$$

Then (3.3) implies that

$$\int f\phi\,dx = A\int f\eta\,dx = c\int \phi\,dx, \qquad c = \int f\eta\,dx.$$

It follows that

$$\int (f-c)\,\phi\,dx = 0 \qquad \text{for all } \phi \in C_c^\infty(0,1),$$

which implies that $f = c$ pointwise almost everywhere, so f is equivalent to a constant function.

Example 3.6. For $a \in \mathbb{R}$, define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{|x|^a}. \tag{3.4}$$

Then f is weakly differentiable if $a + 1 < n$ with weak derivative

$$\partial_i f(x) = -\frac{a}{|x|^{a+1}}\frac{x_i}{|x|}.$$

That is, f is weakly differentiable provided that the pointwise derivative, which is defined almost everywhere, is locally integrable. To prove this, suppose $\epsilon > 0$, and let $\phi^\epsilon \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function that is equal to one in $B_\epsilon(0)$ and zero outside $B_{2\epsilon}(0)$. Then

$$f^\epsilon(x) = \frac{1-\phi^\epsilon(x)}{|x|^a}$$

belongs to $\in C^\infty(\mathbb{R}^n)$ and $f^\epsilon = f$ in $|x| \geq 2\epsilon$. Integrating by parts, we get

$$\int (\partial_i f^\epsilon)\, \phi\, dx = -\int f^\epsilon\, (\partial_i \phi)\, dx.$$

We have

$$\partial_i f^\epsilon(x) = -\frac{a}{|x|^{a+1}} \frac{x_i}{|x|} [1 - \phi^\epsilon(x)] - \frac{1}{|x|^a} \partial_i \phi^\epsilon(x).$$

Since $|\partial_i \phi^\epsilon| \leq C/\epsilon$ and $|\partial_i \phi^\epsilon| = 0$ when $|x| \leq \epsilon$ or $|x| \geq 2\epsilon$, we have

$$|\partial_i \phi^\epsilon(x)| \leq \frac{C}{|x|}.$$

It follows that

$$|\partial_i f^\epsilon(x)| \leq \frac{C'}{|x|^{a+1}}$$

where C' is a constant independent of ϵ. The result then follows from the dominated convergence theorem.

Alternatively, instead of mollifying f, we can use the truncated function

$$f^\epsilon(x) = \frac{\chi_{B_\epsilon(0)}(x)}{|x|^a}.$$

3.3. Distributions

Although we will not make extensive use of the theory of distributions, it is useful to understand the interpretation of a weak derivative as a distributional derivative. In fact, the definition of the weak derivative by Sobolev, and others, was one motivation for the subsequent development of distribution theory by Schwartz.

Let Ω be an open set in $\mathbb{R}^n$.

Definition 3.7. A sequence $\{\phi_n : n \in \mathbb{N}\}$ of functions $\phi_n \in C_c^\infty(\Omega)$ converges to $\phi \in C_c^\infty(\Omega)$ in the sense of test functions if:

(a) there exists $\Omega' \Subset \Omega$ such that $\operatorname{spt} \phi_n \subset \Omega'$ for every $n \in \mathbb{N}$;

(b) $\partial^\alpha \phi_n \to \partial^\alpha \phi$ as $n \to \infty$ uniformly on Ω for every $\alpha \in \mathbb{N}_0^n$.

The topological vector space $\mathcal{D}(\Omega)$ consists of $C_c^\infty(\Omega)$ equipped with the topology that corresponds to convergence in the sense of test functions.

Note that since the supports of the ϕ_n are contained in the same compactly contained subset, the limit has compact support; and since the derivatives of all orders converge uniformly, the limit is smooth.

The space $\mathcal{D}(\Omega)$ is not metrizable, but it can be shown that the sequential convergence of test functions is sufficient to determine its topology.

A linear functional on $\mathcal{D}(\Omega)$ is a linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$. We denote the value of T acting on a test function ϕ by $\langle T, \phi \rangle$; thus, T is linear if

$$\langle T, \lambda\phi + \mu\psi \rangle = \lambda\langle T, \phi \rangle + \mu\langle T, \psi \rangle \qquad \text{for all } \lambda, \mu \in \mathbb{R} \text{ and } \phi, \psi \in \mathcal{D}(\Omega).$$

A functional T is continuous if $\phi_n \to \phi$ in the sense of test functions implies that $\langle T, \phi_n \rangle \to \langle T, \phi \rangle$ in $\mathbb{R}$

DEFINITION 3.8. A distribution on Ω is a continuous linear functional

$$T : \mathcal{D}(\Omega) \to \mathbb{R}.$$

A sequence $\{T_n : n \in \mathbb{N}\}$ of distributions converges to T, written $T_n \rightharpoonup T$, if $\langle T_n, \phi \rangle \to \langle T, \phi \rangle$ for every $\phi \in \mathcal{D}(\Omega)$. The topological vector space $\mathcal{D}'(\Omega)$ consists of the distributions on Ω equipped with the topology corresponding to this notion of convergence.

Thus, the space of distributions is the topological dual of the space of test functions.

EXAMPLE 3.9. The delta-function supported at a is the distribution

$$\delta_a : \mathcal{D}(\Omega) \to \mathbb{R}$$

defined by evaluation of a test function at a:

$$\langle \delta_a, \phi \rangle = \phi(a).$$

This functional is continuous since $\phi_n \to \phi$ in the sense of test functions implies, in particular, that $\phi_n(a) \to \phi(a)$

EXAMPLE 3.10. Any function $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$\langle T_f, \phi \rangle = \int_\Omega f\phi\, dx.$$

The linear functional T_f is continuous since if $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, then

$$\sup_{\Omega'} |\phi_n - \phi| \to 0$$

on a set $\Omega' \Subset \Omega$ that contains the supports of the ϕ_n, so

$$|\langle T, \phi_n \rangle - \langle T, \phi \rangle| = \left| \int_{\Omega'} f\,(\phi_n - \phi)\; dx \right| \le \left(\int_{\Omega'} |f|\, dx \right) \sup_{\Omega'} |\phi_n - \phi| \to 0.$$

Any distribution associated with a locally integrable function in this way is called a regular distribution. We typically regard the function f and the distribution T_f as equivalent.

EXAMPLE 3.11. If μ is a Radon measure on Ω, then

$$\langle I_\mu, \phi \rangle = \int_\Omega \phi\, d\mu$$

defines a distribution $I_\mu \in \mathcal{D}'(\Omega)$. This distribution is regular if and only if μ is locally absolutely continuous with respect to Lebesgue measure λ, in which case the Radon-Nikodym derivative

$$f = \frac{d\mu}{d\lambda} \in L^1_{\mathrm{loc}}(\Omega)$$

is locally integrable, and

$$\langle I_\mu, \phi \rangle = \int_\Omega f\phi\, dx$$

so $I_\mu = T_f$. On the other hand, if μ is singular with respect to Lebesgue measure (for example, if $\mu = \delta_c$ is the unit point measure supported at $c \in \Omega$), then I_μ is not a regular distribution.

One of the main advantages of distributions is that, in contrast to functions, every distribution is differentiable. The space of distributions may be thought of as the smallest extension of the space of continuous functions that is closed under differentiation.

DEFINITION 3.12. For $1 \leq i \leq n$, the ith partial derivative of a distribution $T \in \mathcal{D}'(\Omega)$ is the distribution $\partial_i T \in \mathcal{D}'(\Omega)$ defined by

$$\langle \partial_i T, \phi \rangle = -\langle T, \partial_i \phi \rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

For $\alpha \in \mathbb{N}_0^n$, the derivative $\partial^\alpha T \in \mathcal{D}'(\Omega)$ of order $|\alpha|$ is defined by

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Note that if $T \in \mathcal{D}'(\Omega)$, then it follows from the linearity and continuity of the derivative $\partial^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ on the space of test functions that $\partial^\alpha T$ is a continuous linear functional on $\mathcal{D}(\Omega)$. Thus, $\partial^\alpha T \in \mathcal{D}'(\Omega)$ for any $T \in \mathcal{D}'(\Omega)$. It also follows that the distributional derivative $\partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is linear and continuous on the space of distributions; in particular if $T_n \rightharpoonup T$, then $\partial^\alpha T_n \rightharpoonup \partial^\alpha T$.

Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a locally integrable function and $T_f \in \mathcal{D}'(\Omega)$ the associated regular distribution defined in Example 3.10. Suppose that the distributional derivative of T_f is a regular distribution

$$\partial_i T_f = T_{g_i} \qquad g_i \in L^1_{\mathrm{loc}}(\Omega).$$

Then it follows from the definitions that

$$\int_\Omega f \partial_i \phi \, dx = -\int_\Omega g_i \phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

Thus, Definition 3.1 of the weak derivative may be restated as follows: A locally integrable function is weakly differentiable if its distributional derivative is regular, and its weak derivative is the locally integrable function corresponding to the distributional derivative.

The distributional derivative of a function exists even if the function is not weakly differentiable.

EXAMPLE 3.13. If f is a function of bounded variation, then the distributional derivative of f is a finite Radon measure, which need not be regular. For example, the distributional derivative of the step function is the delta-function, and the distributional derivative of the Cantor function is the corresponding Lebesgue-Stietjes measure supported on the Cantor set.

EXAMPLE 3.14. The derivative of the delta-function δ_a supported at a, defined in Example 3.9, is the distribution $\partial_i \delta_a$ defined by

$$\langle \partial_i \delta_a, \phi \rangle = -\partial_i \phi(a).$$

This distribution is neither regular nor a Radon measure.

Differential equations are typically thought of as equations that relate functions. The use of weak derivatives and distribution theory leads to an alternative point of view of linear differential equations as linear functionals acting on test functions. Using this perspective, given suitable estimates, one can obtain simple and general existence results for weak solutions of linear PDEs by the use of the Hahn-Banach, Riesz representation, or other duality theorems for the existence of bounded linear functionals.

While distribution theory provides an effective general framework for the analysis of linear PDEs, it is less useful for nonlinear PDEs because one cannot define a product of distributions that extends the usual product of smooth functions in an unambiguous way. For example, what is $T_f \delta_a$ if f is a locally integrable function that is discontinuous at a? There are difficulties even for regular distributions. For example, $f : x \mapsto |x|^{-n/2}$ is locally integrable on $\mathbb{R}^n$ but f^2 is not, so how should one define the distribution $(T_f)^2$?

3.4. Properties of weak derivatives

We collect here some properties of weak derivatives. The first result is a product rule.

PROPOSITION 3.15. *If $f \in L^1_{\mathrm{loc}}(\Omega)$ has weak partial derivative $\partial_i f \in L^1_{\mathrm{loc}}(\Omega)$ and $\psi \in C^\infty(\Omega)$, then ψf is weakly differentiable with respect to x_i and*

$$\partial_i(\psi f) = (\partial_i \psi) f + \psi(\partial_i f). \tag{3.5}$$

PROOF. Let $\phi \in C_c^\infty(\Omega)$ be any test function. Then $\psi\phi \in C_c^\infty(\Omega)$ and the weak differentiability of f implies that

$$\int_\Omega f \partial_i(\psi\phi)\, dx = -\int_\Omega (\partial_i f)\psi\phi\, dx.$$

Expanding $\partial_i(\psi\phi) = \psi(\partial_i\phi) + (\partial_i\psi)\phi$ in this equation and rearranging the result, we get

$$\int_\Omega \psi f(\partial_i\phi)\, dx = -\int_\Omega \left[(\partial_i\psi) f + \psi(\partial_i f)\right]\phi\, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

Thus, ψf is weakly differentiable and its weak derivative is given by (3.5). □

The commutativity of weak derivatives follows immediately from the commutativity of derivatives applied to smooth functions.

PROPOSITION 3.16. *Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$ and that the weak derivatives $\partial^\alpha f$, $\partial^\beta f$ exist for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. Then if any one of the weak derivatives $\partial^{\alpha+\beta} f$, $\partial^\alpha\partial^\beta f$, $\partial^\beta\partial^\alpha f$ exists, all three derivatives exist and are equal.*

PROOF. Using the existence of $\partial^\alpha u$, and the fact that $\partial^\beta\phi \in C_c^\infty(\Omega)$ for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \partial^\alpha u \partial^\beta \phi\, dx = (-1)^{|\alpha|}\int_\Omega u\partial^{\alpha+\beta}\phi\, dx.$$

This equation shows that $\partial^{\alpha+\beta}u$ exists if and only if $\partial^\beta\partial^\alpha u$ exists, and in that case the weak derivatives are equal. Using the same argument with α and β exchanged, we get the result. □

EXAMPLE 3.17. Consider functions of the form

$$u(x, y) = f(x) + g(y).$$

Then $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ if and only if $f, g \in L^1_{\mathrm{loc}}(\mathbb{R})$. The weak derivative $\partial_x u$ exists if and only if the weak derivative f' exists, and then $\partial_x u(x, y) = f'(x)$. To see this, we use Fubini's theorem to get for any $\phi \in C_c^\infty(\mathbb{R}^2)$ that

$$\begin{aligned}\int u(x,y)\partial_x\phi(x,y)\,dxdy \\ = \int f(x)\partial_x\left[\int \phi(x,y)\,dy\right]\,dx + \int g(y)\left[\int \partial_x\phi(x,y)\,dx\right]\,dy.\end{aligned}$$

Since ϕ has compact support,

$$\int \partial_x\phi(x,y)\,dx = 0.$$

Also,

$$\int \phi(x,y)\,dy = \xi(x)$$

is a test function $\xi \in C_c^\infty(\mathbb{R})$. Moreover, by taking $\phi(x,y) = \xi(x)\eta(y)$, where $\eta \in C_c^\infty(\mathbb{R})$ is an arbitrary test function with integral equal to one, we can get every $\xi \in C_c^\infty(\mathbb{R})$. Since

$$\int u(x,y)\partial_x\phi(x,y)\,dxdy = \int f(x)\xi'(x)\,dx,$$

it follows that $\partial_x u$ exists if and only if f' exists, and then $\partial_x u = f'$.

In that case, the mixed derivative $\partial_y\partial_x u$ also exists, and is zero, since using Fubini's theorem as before

$$\int f'(x)\partial_y\phi(x,y)\,dxdy = \int f'(x)\left[\int \partial_y\phi(x,y)\,dy\right]\,dx = 0.$$

Similarly $\partial_y u$ exists if and only if g' exists, and then $\partial_y u = g'$ and $\partial_x\partial_y u = 0$.

The second-order weak derivative $\partial_{xy} u$ exists without any differentiability assumptions on $f, g \in L^1_{\mathrm{loc}}(\mathbb{R})$ and is equal to zero. For any $\phi \in C_c^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}\int u(x,y)\partial_{xy}\phi(x,y)\,dxdy \\ = \int f(x)\partial_x\left(\int \partial_y\phi(x,y)\,dy\right)dx + \int g(y)\partial_y\left(\int \partial_x\phi(x,y)\,dx\right)dy \\ = 0.\end{aligned}$$

Thus, the mixed derivatives $\partial_x\partial_y u$ and $\partial_y\partial_x u$ are equal, and are equal to the second-order derivative $\partial_{xy} u$, whenever both are defined.

Weak derivatives combine well with mollifiers. If Ω is an open set in $\mathbb{R}^n$ and $\epsilon > 0$, we define Ω^ϵ as in (1.6) and let η^ϵ be the standard mollifier (1.5).

THEOREM 3.18. *Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$. Then $\eta^\epsilon * f \in C^\infty(\Omega^\epsilon)$ and*

$$\partial^\alpha\left(\eta^\epsilon * f\right) = \eta^\epsilon * \left(\partial^\alpha f\right).$$

Moreover,

$$\partial^\alpha\left(\eta^\epsilon * f\right) \to \partial^\alpha f \qquad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ as } \epsilon \to 0^+.$$

PROOF. From Theorem 1.22, we have $\eta^\epsilon * f \in C^\infty(\Omega^\epsilon)$ and

$$\partial^\alpha\left(\eta^\epsilon * f\right) = \left(\partial^\alpha \eta^\epsilon\right) * f.$$

Using the fact that $y \mapsto \eta^\epsilon(x-y)$ defines a test function in $C_c^\infty(\Omega)$ for any fixed $x \in \Omega^\epsilon$ and the definition of the weak derivative, we have

$$\begin{aligned}
(\partial^\alpha \eta^\epsilon) * f(x) &= \int \partial_x^\alpha \eta^\epsilon(x-y) f(y)\, dy \\
&= (-1)^{|\alpha|} \int \partial_y^\alpha \eta^\epsilon(x-y) f(y) \\
&= \int \eta^\epsilon(x-y) \partial^\alpha f(y)\, dy \\
&= \eta^\epsilon * (\partial^\alpha f)(x)
\end{aligned}$$

Thus $(\partial^\alpha \eta^\epsilon) * f = \eta^\epsilon * (\partial^\alpha f)$. Since $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$, Theorem 1.22 implies that

$$\eta^\epsilon * (\partial^\alpha f) \to \partial^\alpha f$$

in $L^1_{\mathrm{loc}}(\Omega)$, which proves the result. □

The next result gives an alternative way to characterize weak derivatives as limits of derivatives of smooth functions.

THEOREM 3.19. *A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable in Ω with weak derivative $g = \partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ if and only if there is a sequence $\{f_n\}$ of functions $f_n \in C^\infty(\Omega)$ such that $f_n \to f$ and $\partial^\alpha f_n \to g$ in $L^1_{\mathrm{loc}}(\Omega)$.*

PROOF. If f is weakly differentiable, we may construct an appropriate sequence by mollification as in Theorem 3.18. Conversely, suppose that such a sequence exists. Note that if $f_n \to f$ in $L^1_{\mathrm{loc}}(\Omega)$ and $\phi \in C_c(\Omega)$, then

$$\int_\Omega f_n \phi\, dx \to \int_\Omega f\phi\, dx \qquad \text{as } n \to \infty,$$

since if $K = \operatorname{spt} \phi \Subset \Omega$

$$\left| \int_\Omega f_n \phi\, dx - \int_\Omega f\phi\, dx \right| = \left| \int_K (f_n - f)\, \phi\, dx \right| \le \sup_K |\phi| \int_K |f_n - f|\, dx \to 0.$$

Thus, for any $\phi \in C_c^\infty(\Omega)$, the L^1_{loc}-convergence of f_n and $\partial^\alpha f_n$ implies that

$$\begin{aligned}
\int_\Omega f \partial^\alpha \phi\, dx &= \lim_{n\to\infty} \int_\Omega f_n \partial^\alpha \phi\, dx \\
&= (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega \partial^\alpha f_n \phi\, dx \\
&= (-1)^{|\alpha|} \int_\Omega g\phi\, dx.
\end{aligned}$$

So f is weakly differentiable and $\partial^\alpha f = g$. □

We can use this approximation result to derive properties of the weak derivative as a limit of corresponding properties of smooth functions. For example the following weak versions of the product and chain rule, which are not stated in maximum generality, may be derived in this way.

PROPOSITION 3.20. *Let Ω be an open set in $\mathbb{R}^n$.*

(1) *Suppose that $a \in C^1(\Omega)$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable. Then au is weakly differentiable and*

$$\partial_i(au) = a\,(\partial_i u) + (\partial_i a)\, u.$$

(2) *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable function with* $f' \in L^\infty(\mathbb{R})$ *and* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is weakly differentiable. Then* $v = f \circ u$ *is weakly differentiable and*

$$\partial_i v = f'(u)\partial_i u.$$

(3) *Suppose that* $\phi : \Omega \to \widetilde{\Omega}$ *is a* C^1*-diffeomorphism of* Ω *onto* $\widetilde{\Omega} = \phi(\Omega) \subset \mathbb{R}^n$*. For* $u \in L^1_{\mathrm{loc}}(\Omega)$*, define* $v \in L^1_{\mathrm{loc}}(\widetilde{\Omega})$ *by* $v = u \circ \phi^{-1}$*. Then* v *is weakly differentiable in* $\widetilde{\Omega}$ *if and only if* u *is weakly differentiable in* Ω*, and*

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial \phi_j}{\partial x_i} \frac{\partial v}{\partial y_j} \circ \phi.$$

3.5. Sobolev spaces

Sobolev spaces consist of functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of PDEs.

DEFINITION 3.21. Suppose that Ω is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that

$$\partial^\alpha f \in L^p(\Omega) \qquad \text{for } 0 \le |\alpha| \le k.$$

We write $W^{k,2}(\Omega) = H^k(\Omega)$.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha f|^p \, dx \right)^{1/p}$$

for $1 \le p < \infty$ and

$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \le k} \sup_\Omega |\partial^\alpha f| .$$

As usual, we identify functions that are equal almost everywhere. We will use these norms as the standard ones on $W^{k,p}(\Omega)$, but there are other equivalent norms *e.g.*

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \left(\int_\Omega |\partial^\alpha f|^p \, dx \right)^{1/p},$$

$$\|f\|_{W^{k,p}(\Omega)} = \max_{|\alpha| \le k} \left(\int_\Omega |\partial^\alpha f|^p \, dx \right)^{1/p}.$$

The space $H^k(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{|\alpha| \le k} \int_\Omega (\partial^\alpha f)(\partial^\alpha g) \, dx.$$

We will consider the following properties of Sobolev spaces in the simplest settings.

(1) Approximation of Sobolev functions by smooth functions;
(2) Embedding theorems;
(3) Boundary values of Sobolev functions and trace theorems;
(4) Compactness results.

3.6. Approximation of Sobolev functions

To begin with, we consider Sobolev functions defined on all of $\mathbb{R}^n$. They may be approximated in the Sobolev norm by by test functions.

THEOREM 3.22. *For $k \in \mathbb{N}$ and $1 \le p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$*

PROOF. Let $\eta^\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and $f \in W^{k,p}(\mathbb{R}^n)$. Then Theorem 1.22 and Theorem 3.18 imply that $\eta^\epsilon * f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and for $|\alpha| \le k$

$$\partial^\alpha (\eta^\epsilon * f) = \eta^\epsilon * (\partial^\alpha f) \to \partial^\alpha f \qquad \text{in } L^p(\mathbb{R}^n) \text{ as } \epsilon \to 0^+.$$

It follows that $\eta^\epsilon * f \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $\epsilon \to 0$. Therefore $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Now suppose that $f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, and let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Define $\phi^R(x) = \phi(x/R)$ and $f^R = \phi^R f \in C_c^\infty(\mathbb{R}^n)$. Then, by the Leibnitz rule,

$$\partial^\alpha f^R = \phi^R \partial^\alpha f + \frac{1}{R} h^R$$

where h^R is bounded in L^p uniformly in R. Hence, by the dominated convergence theorem

$$\partial^\alpha f^R \to \partial^\alpha f \qquad \text{in } L^p \text{ as } R \to \infty,$$

so $f^R \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $R \to \infty$. It follows that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\mathbb{R}^n)$. $\square$

If Ω is a proper open subset of $\mathbb{R}^n$, then $C_c^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$. Instead, its closure is the space of functions $W_0^{k,p}(\Omega)$ that 'vanish on the boundary $\partial\Omega$.' We discuss this further below. The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for any open set Ω (Meyers and Serrin, 1964), so that $W^{k,p}(\Omega)$ may alternatively be defined as the completion of the space of smooth functions in Ω whose derivatives of order less than or equal to k belong to $L^p(\Omega)$. Such functions need not extend to continuous functions on $\overline{\Omega}$ or be bounded on Ω.

3.7. Sobolev embedding: $p < n$

> G. H. Hardy reported Harald Bohr as saying 'all analysts spend half their time hunter through the literature for inequalities which they want to use but cannot prove.'[2]

Let us first consider the following basic question: Can we estimate the $L^q(\mathbb{R}^n)$-norm of a smooth, compactly supported function in terms of the $L^p(\mathbb{R}^n)$-norm of its derivative? As we will show, given $1 \le p < n$, this is possible for a unique value of q, called the Sobolev conjugate of p.

We may motivate the answer by means of a scaling argument. We are looking for an estimate of the form

$$\|f\|_{L^q} \le C \|Df\|_{L^p} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n) \tag{3.6}$$

[2]From the Introduction of [**9**].

for some constant $C = C(p, q, n)$. For $\lambda > 0$, let f_λ denote the rescaled function

$$f_\lambda(x) = f\left(\frac{x}{\lambda}\right).$$

Then, changing variables $x \mapsto \lambda x$ in the integrals that define the L^p, L^q norms, with $1 \le p, q < \infty$, and using the fact that

$$Df_\lambda = \frac{1}{\lambda}(Df)_\lambda$$

we find that

$$\left(\int_{\mathbb{R}^n} |Df_\lambda|^p \, dx\right)^{1/p} = \lambda^{n/p-1}\left(\int_{\mathbb{R}^n} |Df|^p \, dx\right)^{1/p},$$
$$\left(\int_{\mathbb{R}^n} |f_\lambda|^q \, dx\right)^{1/q} = \lambda^{n/q}\left(\int_{\mathbb{R}^n} |f|^q \, dx\right)^{1/q}.$$

These norms must scale according to the same exponent if we are to have an inequality of the desired form, otherwise we can violate the inequality by taking $\lambda \to 0$ or $\lambda \to \infty$. The equality of exponents implies that $q = p^*$ where p^* satifies

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{3.7}$$

Note that we need $1 \le p < n$ to ensure that $p^* > 0$, in which case $p < p^* < \infty$. We assume that $n \ge 2$. Writing the solution of (3.7) for p^* explicitly, we make the following definition.

Definition 3.23. If $1 \le p < n$, the Sobolev conjugate p^* of p is

$$p^* = \frac{np}{n-p}.$$

Thus, an estimate of the form (3.6) is possible only if $q = p^*$; we will show that (3.6) is, in fact, true when $q = p^*$. This result was obtained by Sobolev (1938), who used potential-theoretic methods. The proof we give is due to Nirenberg (1959). Before describing the proof, we introduce some notation, explain the main idea, and establish a preliminary inequality.

For $1 \le i \le n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let

$$x_i' = (x_1, \dots, \hat{x}_i, \dots x_n) \in \mathbb{R}^{n-1},$$

where the 'hat' means that the ith coordinate is omitted. We write $x = (x_i, x_i')$ and denote the value of a function $f : \mathbb{R}^n \to \mathbb{R}$ at x by

$$f(x) = f\left(x_i, x_i'\right).$$

We denote the partial derivative with respect to x_i by ∂_i.

If f is smooth with compact support, the fundamental theorem of calculus implies that

$$f(x) = \int_{-\infty}^{x_i} \partial_i f(t, x_i')\, dt.$$

Taking absolute values, we get

$$|f(x)| \le \int_{-\infty}^{\infty} |\partial_i f(t, x_i')| \, dt.$$

We can improve the constant in this estimate by using the fact that

$$\int_{-\infty}^{\infty} \partial_i f(t, x_i')\, dt = 0.$$

LEMMA 3.24. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is an integrable function with compact support such that $\int g\, dt = 0$. If*

$$f(x) = \int_{-\infty}^{x} g(t)\, dt,$$

then

$$|f(x)| \leq \frac{1}{2} \int |g|\, dt.$$

PROOF. Let $g = g_+ - g_-$ where the nonnegative functions g_+, g_- are defined by $g_+ = \max(g, 0)$, $g_- = \max(-g, 0)$. Then $|g| = g_+ + g_-$ and

$$\int g_+\, dt = \int g_-\, dt = \frac{1}{2} \int |g|\, dt.$$

It follows that

$$\begin{aligned} f(x) &\leq \int_{-\infty}^{x} g_+(t)\, dt \leq \int_{-\infty}^{\infty} g_+(t)\, dt \leq \frac{1}{2} \int |g|\, dt, \\ f(x) &\geq -\int_{-\infty}^{x} g_-(t)\, dt \geq -\int_{-\infty}^{\infty} g_-(t)\, dt \geq -\frac{1}{2} \int |g|\, dt, \end{aligned}$$

which proves the result. □

Thus, for $1 \leq i \leq n$ we have

$$|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |\partial_i f(t, x_i')|\, dt.$$

The idea of the proof is to average a suitable power of this inequality over the i-directions and integrate the result to estimate f in terms of Df. In order to do this, we use the following inequality, which estimates the L^1-norm of a function of $x \in \mathbb{R}^n$ in terms of the L^{n-1}-norms of n functions of $x_i' \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

THEOREM 3.25. *Suppose that $n \geq 2$ and*

$$\left\{ g_i \in C_c^\infty(\mathbb{R}^{n-1}) : 1 \leq i \leq n \right\}$$

are nonnegative functions. Define $g \in C_c^\infty(\mathbb{R}^n)$ by

$$g(x) = \prod_{i=1}^{n} g_i(x_i').$$

Then

$$\int g\, dx \leq \prod_{i=1}^{n} \|g_i\|_{n-1}\,. \tag{3.8}$$

Before proving the theorem, we consider what it says in more detail. If $n = 2$, the theorem states that

$$\int g_1(x_2) g_2(x_1)\, dx_1 dx_2 \leq \left(\int g_1(x_2)\, dx_2 \right) \left(\int g_2(x_1)\, dx_1 \right),$$

which follows immediately from Fubini's theorem. If $n = 3$, the theorem states that

$$\int g_1(x_2, x_3) g_2(x_1, x_3) g_3(x_1, x_2)\, dx_1 dx_2 dx_3$$
$$\leq \left(\int g_1^2(x_2, x_3)\, dx_2 dx_3\right)^{1/2} \left(\int g_2^2(x_1, x_3)\, dx_1 dx_3\right)^{1/2} \left(\int g_3^2(x_1, x_2)\, dx_1 dx_2\right)^{1/2}.$$

To prove the inequality in this case, we fix x_1 and apply the Cauchy-Schwartz inequality to the x_2x_3-integral of $g_1 \cdot g_2 g_3$. We then use the inequality for $n = 2$ to estimate the x_2x_3-integral of $g_2 g_3$, and integrate the result over x_1. An analogous approach works for higher n.

Note that under the scaling $g_i \mapsto \lambda g_i$, both sides of (3.8) scale in the same way,

$$\int g\, dx \mapsto \left(\prod_{i=1}^{n} \lambda_i\right) \int g\, dx, \qquad \prod_{i=1}^{n} \|g_i\|_{n-1} \mapsto \left(\prod_{i=1}^{n} \lambda_i\right) \prod_{i=1}^{n} \|g_i\|_{n-1}$$

as must be true for any inequality involving norms. Also, under the spatial rescaling $x \mapsto \lambda x$, we have

$$\int g\, dx \mapsto \lambda^{-n} \int g\, dx,$$

while $\|g_i\|_p \mapsto \lambda^{-(n-1)/p} \|g_i\|_p$, so

$$\prod_{i=1}^{n} \|g_i\|_p \mapsto \lambda^{-n(n-1)/p} \prod_{i=1}^{n} \|g_i\|_p$$

Thus, if $p = n - 1$ the two terms scale in the same way, which explains the appearance of the L^{n-1}-norms of the g_i's on the right hand side of (3.8).

Proof. We use proof by induction. The result is true when $n = 2$. Suppose that it is true for $n - 1$ where $n \geq 3$.

For $1 \leq i \leq n$, let $g_i : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g\ : \mathbb{R}^n \to \mathbb{R}$ be the functions given in the theorem. Fix $x_1 \in \mathbb{R}$ and define $g_{x_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{x_1}(x_1') = g(x_1, x_1').$$

For $2 \leq i \leq n$, let $x_i' = \left(x_1, x_{1,i}'\right)$ where

$$x_{1,i}' = (\hat{x}_1, \ldots, \hat{x}_i, \ldots x_n) \in \mathbb{R}^{n-2}.$$

Define $g_{i,x_1} : \mathbb{R}^{n-2} \to \mathbb{R}$ and $\tilde{g}_{i,x_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{i,x_1}\left(x_{1,i}'\right) = g_i\left(x_1, x_{1,i}'\right).$$

Then

$$g_{x_1}(x_1') = g_1(x_1') \prod_{i=2}^{n} g_{i,x_1}\left(x_{1,i}'\right).$$

Using Hölder's inequality with $q = n - 1$ and $q' = (n-1)/(n-2)$, we get

$$\int g_{x_1}\, dx_1' = \int g_1 \left(\prod_{i=2}^{n} g_{i,x_1}\left(x_{1,i}'\right)\right) dx_1'$$
$$\leq \|g_1\|_{n-1} \left[\int \left(\prod_{i=2}^{n} g_{i,x_1}\left(x_{1,i}'\right)\right)^{(n-1)/(n-2)} dx_1'\right]^{(n-2)/(n-1)}.$$

The induction hypothesis implies that

$$\int \left(\prod_{i=2}^{n} g_{i,x_1}\left(x'_{1,i}\right)\right)^{(n-1)/(n-2)} dx'_1 \le \prod_{i=2}^{n} \left\| g_{i,x_1}^{(n-1)/(n-2)} \right\|_{n-2} \le \prod_{i=2}^{n} \|g_{i,x_1}\|_{n-1}^{(n-1)/(n-2)}.$$

Hence,

$$\int g_{x_1}\, dx'_1 \le \|g_1\|_{n-1} \prod_{i=2}^{n} \|g_{i,x_1}\|_{n-1}.$$

Integrating this equation over x_1 and using the generalized Hölder inequality with $p_2 = p_3 = \cdots = p_n = n-1$, we get

$$\begin{aligned}\int g\, dx &\le \|g_1\|_{n-1} \int \left(\prod_{i=2}^{n} \|g_{i,x_1}\|_{n-1}\right) dx_1 \\ &\le \|g_1\|_{n-1} \left(\prod_{i=2}^{n} \int \|g_{i,x_1}\|_{n-1}^{n-1}\, dx_1\right)^{1/(n-1)}.\end{aligned}$$

Thus, since

$$\begin{aligned}\int \|g_{i,x_1}\|_{n-1}^{n-1}\, dx_1 &= \int \left(\int \left|g_{i,x_1}(x'_{1,i})\right|^{n-1} dx'_{1,i}\right) dx_1 \\ &= \int |g_i(x'_i)|^{n-1}\, dx'_i \\ &= \|g_i\|_{n-1}^{n-1},\end{aligned}$$

we find that

$$\int g\, dx \le \prod_{i=1}^{n} \|g_i\|_{n-1}.$$

The result follows by induction. □

We now prove the main result.

THEOREM 3.26. *Let $1 \le p < n$, where $n \ge 2$, and let p^* be the Sobolev conjugate of p given in Definition 3.23. Then*

$$\|f\|_{p^*} \le C\, \|Df\|_p, \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n)$$

where

$$C(n,p) = \frac{p}{2n}\left(\frac{n-1}{n-p}\right). \tag{3.9}$$

PROOF. First, we prove the result for $p = 1$. For $1 \le i \le n$, we have

$$|f(x)| \le \frac{1}{2} \int |\partial_i f(t, x'_i)|\, dt.$$

Multiplying these inequalities and taking the $(n-1)$th root, we get

$$|f|^{n/(n-1)} \le \frac{1}{2^{n/(n-1)}}\, g, \qquad g = \prod_{i=1}^{n} \tilde{g}_i$$

where $\tilde{g}_i(x) = g_i(x_i')$ with

$$g_i(x_i') = \left(\int |\partial_i f(t, x_i')| \, dt \right)^{1/(n-1)}.$$

Theorem 3.25 implies that

$$\int g \, dx \leq \prod_{i=1}^{n} \|g_i\|_{n-1}.$$

Since

$$\|g_i\|_{n-1} = \left(\int |\partial_i f| \, dx \right)^{1/(n-1)}$$

it follows that

$$\int |f|^{n/(n-1)} \, dx \leq \frac{1}{2^{n/(n-1)}} \left(\prod_{i=1}^{n} \int |\partial_i f| \, dx \right)^{1/(n-1)}.$$

Note that $n/(n-1) = 1^*$ is the Sobolev conjugate of 1.

Using the arithmetic-geometric mean inequality,

$$\left(\prod_{i=1}^{n} a_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i,$$

we get

$$\int |f|^{n/(n-1)} \, dx \leq \left(\frac{1}{2n} \sum_{i=1}^{n} \int |\partial_i f| \, dx \right)^{n/(n-1)},$$

or

$$\|f\|_{1^*} \leq \frac{1}{2n} \|Df\|_1,$$

which proves the result when $p = 1$.

Next suppose that $1 < p < n$. For any $s > 1$, we have

$$\frac{d}{dx} |x|^s = s \operatorname{sgn} x |x|^{s-1}.$$

Thus,

$$\begin{aligned} |f(x)|^s &= \int_{-\infty}^{x_i} \partial_i |f(t, x_i')|^s \, dt \\ &= s \int_{-\infty}^{x_i} |f(t, x_i')|^{s-1} \operatorname{sgn}[f(t, x_i')] \, \partial_i f(t, x_i') \, dt. \end{aligned}$$

Using Lemma 3.24, it follows that

$$|f(x)|^s \leq \frac{s}{2} \int_{-\infty}^{\infty} \left| f^{s-1}(t, x_i') \partial_i f(t, x_i') \right| \, dt,$$

and multiplication of these inequalities gives

$$|f(x)|^{sn} \leq \left(\frac{s}{2} \right)^n \prod_{i=1}^{n} \int_{-\infty}^{\infty} \left| f^{s-1}(t, x_i') \partial_i f(t, x_i') \right| \, dt.$$

Applying Theorem 3.25 with the functions

$$g_i(x_i') = \left[\int_{-\infty}^{\infty} \left| f^{s-1}(t, x_i') \partial_i f(t, x_i') \right| \, dt \right]^{1/(n-1)}$$

we find that

$$\|f\|_{sn/(n-1)}^{sn} \leq \frac{s}{2} \prod_{i=1}^{n} \left\|f^{s-1}\partial_i f\right\|_1 .$$

From Hölder's inequality,

$$\left\|f^{s-1}\partial_i f\right\|_1 \leq \left\|f^{s-1}\right\|_{p'} \|\partial_i f\|_p .$$

We have

$$\left\|f^{s-1}\right\|_{p'} = \|f\|_{p'(s-1)}^{s-1}$$

We choose $s > 1$ so that

$$p'(s-1) = \frac{sn}{n-1},$$

which holds if

$$s = p\left(\frac{n-1}{n-p}\right), \qquad \frac{sn}{n-1} = p^*.$$

Then

$$\|f\|_{p^*} \leq \frac{s}{2} \left(\prod_{i=1}^{n} \|\partial_i f\|_p\right)^{1/n} .$$

Using the arithmetic-geometric mean inequality, we get

$$\|f\|_{p^*} \leq \frac{s}{2n} \left(\sum_{i=1}^{n} \|\partial_i f\|_p^p\right)^{1/p} ,$$

which proves the result. □

We can interpret this result roughly as follows: Differentiation of a function increases the strength of its local singularities and improves its decay at infinity. Thus, if $Df \in L^p$, it is reasonable to expect that $f \in L^{p^*}$ for some $p^* > p$ since L^{p^*}-functions have weaker singularities and can decay more slowly at infinity than L^p-functions.

Example 3.27. For $a > 0$, let $f_a : \mathbb{R}^n \to \mathbb{R}$ be the function

$$f_a(x) = \frac{1}{|x|^a}$$

considered in Example 3.6. This function does not belong to $L^q(\mathbb{R}^n)$ for any a since the integral at infinity diverges whenever the integral at zero converges. Let ϕ be a smooth cut-off function that is equal to one for $|x| \leq 1$ and zero for $|x| \geq 2$. Then $g_a = \phi f_a$ is an unbounded function with compact support. We have $g_a \in L^q(\mathbb{R}^n)$ if $aq < n$, and $Dg_a \in L^p(\mathbb{R}^n)$ if $p(a+1) < n$ or $ap^* < n$. Thus if $Dg_a \in L^p(\mathbb{R}^n)$, then $g_a \in L^q(\mathbb{R}^n)$ for $1 \leq q \leq p^*$. On the other hand, the function $h_a = (1-\phi)f_a$ is smooth and decays like $|x|^{-a}$ as $x \to \infty$. We have $h_a \in L^q(\mathbb{R}^n)$ if $qa > n$ and $Dh_a \in L^p(\mathbb{R}^n)$ if $p(a+1) > n$ or $p^*a > n$. Thus, if $Dh_a \in L^p(\mathbb{R}^n)$, then $f \in L^q(\mathbb{R}^n)$ for $p^* \leq q < \infty$. The function $f_{ab} = g_a + h_b$ belongs to $L^{p^*}(\mathbb{R}^n)$ for any choice of $a, b > 0$ such that $Df_{ab} \in L^p(\mathbb{R}^n)$. On the other hand, for any $1 \leq q \leq \infty$ such that $q \neq p^*$, there is a choice of $a, b > 0$ such that $Df_{ab} \in L^p(\mathbb{R}^n)$ but $f_{ab} \notin L^q(\mathbb{R}^n)$.

The constant in Theorem 3.26 is not optimal. For $p = 1$, the best constant is

$$C(n,1) = \frac{1}{n\alpha_n^{1/n}}$$

where α_n is the volume of the unit ball, or

$$C(n,1) = \frac{1}{n\sqrt{\pi}} \left[\Gamma\left(1 + \frac{n}{2}\right)\right]^{1/n}$$

where Γ is the Γ-function. Equality is obtained in the limit of functions that approach the characteristic function of a ball. This result for the best Sobolev constant is equivalent to the isoperimetric inequality that a sphere has minimal area among all surfaces enclosing a given volume.

For $1 < p < n$, the best constant is (Talenti, 1976)

$$C(n,p) = \frac{1}{n^{1/p}\sqrt{\pi}} \left(\frac{p-1}{n-p}\right)^{1-1/p} \left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right]^{1/n}.$$

Equality holds for functions of the form

$$f(x) = \left(a + b|x|^{p/(p-1)}\right)^{1-n/p}$$

where a, b are positive constants.

The Sobolev inequality in Theorem 3.26 does not hold in the limiting case $p \to n$, $p^* \to \infty$.

Example 3.28. If $\phi(x)$ is a smooth cut-off function that is equal to one for $|x| \leq 1$ and zero for $|x| \geq 2$, and

$$f(x) = \phi(x) \log\log\left(1 + \frac{1}{|x|}\right),$$

then $Df \in L^n(\mathbb{R}^n)$, but $f \notin L^\infty(\mathbb{R}^n)$.

We can use the Sobolev inequality to prove various embedding theorems. In general, we say that a Banach space X is continuously embedded, or embedded for short, in a Banach space Y if there is a one-to-one, bounded linear map $\imath : X \to Y$. We often think of $\imath$ as identifying elements of the smaller space X with elements of the larger space Y; if X is a subset of Y, then $\imath$ is the inclusion map. The boundedness of $\imath$ means that there is a constant C such that $\|\imath x\|_Y \leq C\|x\|_X$ for all $x \in X$, so the weaker Y-norm of $\imath x$ is controlled by the stronger X-norm of x.

We write an embedding as $X \hookrightarrow Y$, or as $X \subset Y$ when the boundedness is understood.

Theorem 3.29. *Suppose that $1 \leq p < n$ and $p \leq q \leq p^*$ where p^* is the Sobolev conjugate of p. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and*

$$\|f\|_q \leq C\|f\|_{W^{1,p}} \qquad \text{for all } f \in W^{1,p}(\mathbb{R}^n)$$

for some constant $C = C(n,p,q)$.

Proof. If $f \in W^{1,p}(\mathbb{R}^n)$, then by Theorem 3.22 there is a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^n)$ that converges to f in $W^{1,p}(\mathbb{R}^n)$. Theorem 3.26 implies that $f_n \to f$ in $L^{p^*}(\mathbb{R}^n)$. In detail: $\{Df_n\}$ converges to Df in L^p so it is Cauchy in L^p; since

$$\|f_n - f_m\|_{p^*} \leq C\|Df_n - Df_m\|_p$$

$\{f_n\}$ is Cauchy in L^{p^*}; therefore $f_n \to \tilde{f}$ for some $\tilde{f} \in L^{p^*}$ since L^{p^*} is complete; and $\tilde{f}$ is equivalent to f since a subsequence of $\{f_n\}$ converges pointwise a.e. to $\tilde{f}$, from the L^{p^*} convergence, and to f, from the L^p-convergence.

Thus, $f \in L^{p^*}(\mathbb{R}^n)$ and

$$\|f\|_{p^*} \leq C\|Df\|_p.$$

Since $f \in L^p(\mathbb{R}^n)$, Lemma 1.10 implies that for $p < q < p^*$

$$\|f\|_q \leq \|f\|_p^\theta \|f\|_{p^*}^{1-\theta}$$

where $0 < \theta < 1$ is defined by

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}.$$

Therefore, using Theorem 3.26 and the inequality

$$a^\theta b^{1-\theta} \leq \left[\theta^\theta (1-\theta)^{1-\theta}\right]^{1/p} (a^p + b^p)^{1/p},$$

we get

$$\begin{aligned}
\|f\|_q &\leq C^{1-\theta}\|f\|_p^\theta \|Df\|_p^{1-\theta} \\
&\leq C^{1-\theta}\left[\theta^\theta (1-\theta)^{1-\theta}\right]^{1/p} \left(\|f\|_p^p + \|Df\|_p^p\right)^{1/p} \\
&\leq C^{1-\theta}\left[\theta^\theta (1-\theta)^{1-\theta}\right]^{1/p} \|f\|_{W^{1,p}}.
\end{aligned}$$

□

Sobolev embedding gives a stronger conclusion for sets Ω with finite measure. In that case, $L^{p^*}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq p^*$, so $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p^*$, not just $p \leq q \leq p^*$.

Theorem 3.26 does not, of course, imply that $f \in L^{p^*}(\mathbb{R}^n)$ whenever $Df \in L^p(\mathbb{R}^n)$, since constant functions have zero derivative. To ensure that $f \in L^{p^*}(\mathbb{R}^n)$, we also need to impose a decay condition on f that eliminates the constant functions. In Theorem 3.29, this is provided by the assumption that $f \in L^p(\mathbb{R}^n)$ in addition to $Df \in L^p(\mathbb{R}^n)$. The weakest decay condition we can impose is the following one.

Definition 3.30. A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ vanishes at infinity if for every $\epsilon > 0$ the set $\{x \in \mathbb{R}^n : |f(x)| > \epsilon\}$ has finite Lebesgue measure.

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then f vanishes at infinity. Note that this does not imply that $\lim_{|x|\to\infty} f(x) = 0$.

Example 3.31. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f = \sum_{n\in\mathbb{N}} \chi_{I_n}, \qquad I_n = \left[n, n + \frac{1}{n^2}\right]$$

where χ_I is the characteristic function of the interval I. Then

$$\int f\, dx = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty,$$

so $f \in L^1(\mathbb{R})$. The limit of $f(x)$ as $|x| \to \infty$ does not exist since $f(x)$ takes on the values 0 and 1 for arbitrarily large values of x. Nevertheless, f vanishes at infinity since for any $\epsilon < 1$,

$$|\{x \in \mathbb{R} : |f(x)| > \epsilon\}| = \sum_{n\in\mathbb{N}} \frac{1}{n^2},$$

which is finite.

EXAMPLE 3.32. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/\log x & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases}$$

vanishes at infinity, but $f \notin L^p(\mathbb{R})$ for any $1 \leq p < \infty$.

The Sobolev embedding theorem remains true for functions that vanish at infinity.

THEOREM 3.33. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is weakly differentiable with $Df \in L^p(\mathbb{R}^n)$ where $1 \leq p < n$ and f vanishes at infinity. Then $f \in L^{p^*}(\mathbb{R}^n)$ and*

$$\|f\|_{p^*} \leq C\|Df\|_p$$

where C is given in (3.9).

As before, we prove this by approximating f with smooth compactly supported functions. We omit the details.

3.8. Sobolev embedding: $p > n$

In the previous section, we saw that if the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. Even though the function is weakly differentiable, it need not be continuous. In this section, we show that if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise a.e. equivalent version of it) is continuous, and in fact Hölder continuous. The following result is due to Morrey (1940). The main idea is to estimate the difference $|f(x) - f(y)|$ in terms of Df by the mean value theorem, average the result over a ball $B_r(x)$ and estimate the result in terms of $\|Df\|_p$ by Hölder's inequality.

THEOREM 3.34. *Let $n < p < \infty$ and*

$$\alpha = 1 - \frac{n}{p},$$

with $\alpha = 1$ if $p = \infty$. Then there are constants $C = C(n, p)$ such that

$$[f]_\alpha \leq C\,\|Df\|_p \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{3.10}$$

$$\sup_{\mathbb{R}^n} |f| \leq C\,\|f\|_{W^{1,p}} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{3.11}$$

where $[\cdot]_\alpha$ denotes the Hölder seminorm $[\cdot]_{\alpha,\mathbb{R}^n}$ defined in (1.1).

PROOF. First we prove that there exists a constant C depending only on n such that for any ball $B_r(x)$

$$\fint_{B_r(x)} |f(x) - f(y)|\,dy \leq C \int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\,dy \tag{3.12}$$

Let $w \in \partial B_1(0)$ be a unit vector. For $s > 0$

$$f(x + sw) - f(x) = \int_0^s \frac{d}{dt} f(x + tw)\,dt = \int_0^s Df(x + tw) \cdot w\,dt,$$

and therefore since $|w| = 1$

$$|f(x + sw) - f(x)| \leq \int_0^s |Df(x + tw)|\,dt.$$

Integrating this inequality with respect to w over the unit sphere, we get

$$\int_{\partial B_1(0)} |f(x) - f(x+sw)|\, dS(w) \le \int_{\partial B_1(0)} \left(\int_0^s |Df(x+tw)|\, dt\right) dS(w).$$

From Proposition 1.39,

$$\begin{aligned}\int_{\partial B_1(0)} \left(\int_0^s |Df(x+tw)|\, dt\right) dS(w) &= \int_{\partial B_1(0)} \int_0^s \frac{|Df(x+tw)|}{t^{n-1}}\, t^{n-1}\, dt dS(w)\\ &= \int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy,\end{aligned}$$

Thus,

$$\int_{\partial B_1(0)} |f(x) - f(x+sw)|\, dS(w) \le \int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy.$$

Using Proposition 1.39 together with this inequality, and estimating the integral over $B_s(x)$ by the integral over $B_r(x)$ for $s \le r$, we find that

$$\begin{aligned}\int_{B_r(x)} |f(x) - f(y)|\, dy &= \int_0^r \left(\int_{\partial B_1(0)} |f(x) - f(x+sw)|\, dS(w)\right) s^{n-1}\, ds\\ &\le \int_0^r \left(\int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy\right) s^{n-1}\, ds\\ &\le \left(\int_0^r s^{n-1}\, ds\right)\left(\int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy\right)\\ &\le \frac{r^n}{n}\int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy\end{aligned}$$

This gives (3.12) with $C = (n\alpha_n)^{-1}$.

Next, we prove (3.10). Suppose that $x, y \in \mathbb{R}^n$. Let $r = |x-y|$ and $\Omega = B_r(x) \cap B_r(y)$. Then averaging the inequality

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(y) - f(z)|$$

with respect to z over Ω, we get

$$|f(x) - f(y)| \le \fint_\Omega |f(x) - f(z)|\, dz + \fint_\Omega |f(y) - f(z)|\, dz. \tag{3.13}$$

From (3.12) and Hölder's inequality,

$$\begin{aligned}\fint_\Omega |f(x) - f(z)|\, dz &\le \fint_{B_r(x)} |f(x) - f(z)|\, dz\\ &\le C\int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy\\ &\le C\left(\int_{B_r(x)} |Df|^p\, dz\right)^{1/p}\left(\int_{B_r(x)} \frac{dz}{|x-z|^{p'(n-1)}}\right)^{1/p'}.\end{aligned}$$

We have

$$\left(\int_{B_r(x)} \frac{dz}{|x-z|^{p'(n-1)}}\right)^{1/p'} = C\left(\int_0^r \frac{r^{n-1}dr}{r^{p'(n-1)}}\right)^{1/p'} = Cr^{1-n/p}$$

where C denotes a generic constant depending on n and p. Thus,

$$\fint_{\Omega} |f(x) - f(z)| \, dz \leq C r^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)},$$

with a similar estimate for the integral in which x is replaced by y. Using these estimates in (3.13) and setting $r = |x - y|$, we get

$$|f(x) - f(y)| \leq C|x - y|^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}, \tag{3.14}$$

which proves (3.10).

Finally, we prove (3.11). For any $x \in \mathbb{R}^n$, using (3.14), we find that

$$\begin{aligned} |f(x)| &\leq \fint_{B_1(x)} |f(x) - f(y)| \, dy + \fint_{B_1(x)} |f(y)| \, dy \\ &\leq C \|Df\|_{L^p(\mathbb{R}^n)} + C \|f\|_{L^p(B_1(x))} \\ &\leq C \|f\|_{W^{1,p}(\mathbb{R}^n)}, \end{aligned}$$

and taking the supremum with respect to x, we get (3.11). □

Combining these estimates for

$$\|f\|_{C^{0,\alpha}} = \sup |f| + [f]_\alpha$$

and using a density argument, we get the following theorem. We denote by $C_0^{0,\alpha}(\mathbb{R}^n)$ the space of Hölder continuous functions f whose limit as $x \to \infty$ is zero, meaning that for every $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f(x)| < \epsilon$ if $x \in \mathbb{R}^n \setminus K$.

Theorem 3.35. *Let $n < p < \infty$ and $\alpha = 1 - n/p$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$$

and there is a constant $C = C(n,p)$ such that

$$\|f\|_{C^{0,\alpha}} \leq C \|f\|_{W^{1,p}} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

Proof. From Theorem 3.22, the mollified functions $\eta^\epsilon * f^\epsilon \to f$ in $W^{1,p}(\mathbb{R}^n)$ as $\epsilon \to 0^+$, and by Theorem 3.34

$$|f^\epsilon(x) - f^\epsilon(y)| \leq C|x - y|^{1-n/p} \|Df^\epsilon\|_{L^p}.$$

Letting $\epsilon \to 0^+$, we find that

$$|f(x) - f(y)| \leq C|x - y|^{1-n/p} \|Df\|_{L^p}$$

for all Lebesgue points $x, y \in \mathbb{R}^n$ of f. Since these form a set of measure zero, f extends by uniform continuity to a uniformly continuous function on $\mathbb{R}^n$.

Also from Theorem 3.22, the function $f \in W^{1,p}(\mathbb{R}^n)$ is a limit of compactly supported functions, and from (3.11), f is the uniform limit of compactly supported functions, which implies that its limit as $x \to \infty$ is zero. □

We state two results without proof (see §5.8 of [**5**]).

For $p = \infty$, the same proof as the proof of (3.10), using Hölder's inequality with $p = \infty$ and $p' = 1$, shows that $f \in W^{1,\infty}(\mathbb{R}^n)$ is Lipschitz continuous, with

$$[f]_1 \leq C \|Df\|_{L^\infty}.$$

A function in $W^{1,\infty}(\mathbb{R}^n)$ need not approach zero at infinity. We have in this case the following characterization of Lipschitz functions.

THEOREM 3.36. *A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is Lipschitz continuous if and only if it is weakly differentiable and $Df \in L^\infty(\mathbb{R}^n)$.*

When $n < p \leq \infty$, the above estimates can be used to prove that pointwise derivative of a Sobolev function exists almost everywhere and agrees with the weak derivative.

THEOREM 3.37. *If $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ for some $n < p \leq \infty$, then f is differentiable pointwise a.e. and the pointwise derivative coincides with the weak derivative.*

3.9. Boundary values of Sobolev functions

If $f \in C(\overline{\Omega})$ is a continuous function on the closure of a smooth domain Ω, we can define the boundary values of f pointwise as a continuous function on the boundary $\partial\Omega$. We can also do this when Sobolev embedding implies that a function is Hölder continuous. In general, however, a Sobolev function is not equivalent pointwise a.e. to a continuous function and the boundary of a smooth open set has measure zero, so the boundary values cannot be defined pointwise. For example, we cannot make sense of the boundary values of an L^p-function as an L^p-function on the boundary.

EXAMPLE 3.38. Suppose $T : C^\infty([0,1]) \to \mathbb{R}$ is the map defined by $T : \phi \mapsto \phi(0)$. If $\phi^\epsilon(x) = e^{-x^2/\epsilon}$, then $\|\phi^\epsilon\|_{L^1} \to 0$ as $\epsilon \to 0^+$, but $\phi^\epsilon(0) = 1$ for every $\epsilon > 0$. Thus, T is not bounded (or even closed) and we cannot extend it by continuity to $L^1(0,1)$.

Nevertheless, we can define the boundary values of a Sobolev function at the expense of a loss of smoothness in restricting the function to the boundary. To do this, we show that the linear map on smooth functions that gives their boundary values is bounded with respect to appropriate Sobolev norms. We then extend the map by continuity to Sobolev functions, and the resulting trace map defines their boundary values.

We consider the basic case of a half-space $\mathbb{R}^n_+$. We write $x = (x', x_n) \in \mathbb{R}^n_+$ where $x_n > 0$ and $(x', 0) \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$.

The Sobolev space $W^{1,p}(\mathbb{R}^n_+)$ consists of functions $f \in L^p(\mathbb{R}^n_+)$ that are weakly differentiable in $\mathbb{R}^n_+$ with $Df \in L^p(\mathbb{R}^n_+)$. We begin with a result which states that we can extend functions $f \in W^{1,p}(\mathbb{R}^n_+)$ to functions in $W^{1,p}(\mathbb{R}^n)$ without increasing their norm. An extension may be constructed by reflecting a function across the boundary $\partial\mathbb{R}^n_+$ in a way that preserves its differentiability. Such an extension map E is not, of course, unique.

THEOREM 3.39. *There is a bounded linear map*

$$E : W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$$

such that $Ef = f$ pointwise a.e. in $\mathbb{R}^n_+$ and for some constant $C = C(n,p)$

$$\|Ef\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|f\|_{W^{1,p}(\mathbb{R}^n_+)}.$$

The following approximation result may be proved by extending a Sobolev function from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, mollifying the extension, and restricting the result to the half-space.

THEOREM 3.40. *The space $C_c^\infty(\overline{\mathbb{R}}^n_+)$ of smooth functions is dense in $W^{k,p}(\mathbb{R}^n_+)$.*

Functions $f : \overline{\mathbb{R}}^n_+ \to \mathbb{R}$ in $C_c^\infty(\overline{\mathbb{R}}^n_+)$ need not vanish on the boundary $\partial\mathbb{R}^n_+$. On the other hand, functions in the space $C_c^\infty(\mathbb{R}^n_+)$ of smooth functions whose support is contained in the open half space $\mathbb{R}^n_+$ do vanish on the boundary, and it is not true that this space is dense in $W^{k,p}(\mathbb{R}^n_+)$. Roughly speaking, we can only approximate by functions in $C_c^\infty(\mathbb{R}^n_+)$ Sobolev functions that 'vanish on the boundary'. We make the following definition.

Definition 3.41. The space $W_0^{k,p}(\mathbb{R}^n_+)$ is the closure of $C_c^\infty(\mathbb{R}^n_+)$ in $W^{k,p}(\mathbb{R}^n_+)$.

The interpretation of $W_0^{1,p}(\mathbb{R}^n_+)$ as the space of Sobolev functions that vanish on the boundary is made more precise in the following theorem, which shows the existence of a trace map T that maps a Sobolev function to its boundary values, and states that functions in $W_0^{1,p}(\mathbb{R}^n_+)$ are the ones whose trace is equal to zero.

Theorem 3.42. *For $1 \le p < \infty$, there is a bounded linear operator*

$$T : W^{1,p}(\mathbb{R}^n_+) \to L^p(\partial\mathbb{R}^n_+)$$

such that for any $f \in C_c^\infty(\overline{\mathbb{R}}^n_+)$

$$(Tf)(x') = f(x', 0)$$

and

$$\|Tf\|_{L^p(\mathbb{R}^{n-1})} \le C\|f\|_{W^{1,p}(\mathbb{R}^n_+)}$$

for some constant C depending only on p. Furthermore, $f \in W_0^{k,p}(\mathbb{R}^n_+)$ if and only if $Tf = 0$.

Proof. First, we consider $f \in C_c^\infty(\overline{\mathbb{R}}^n_+)$. For $x' \in \mathbb{R}^{n-1}$ and $p \ge 1$, we have

$$|f(x', 0)|^p \le p \int_0^\infty |f(x', t)|^{p-1} |\partial_n f(x', t)|\, dt.$$

Hence, using Hölder's inequality and the identity $p'(p-1) = p$, we get

$$\begin{aligned}
\int |f(x', 0)|^p\, dx' &\le p \int_0^\infty |f(x', t)|^{p-1} |\partial_n f(x', t)|\, dx'dt \\
&\le p \left(\int_0^\infty |f(x', t)|^{p'(p-1)}\, dx'dt \right)^{1/p'} \left(\int_0^\infty |\partial_n f(x', t)|^p\, dx'dt \right)^{1/p} \\
&\le p\|f\|_p^{p-1} \|\partial_n f\|_p \\
&\le p\|f\|^p_{W^{k,p}}.
\end{aligned}$$

The trace map

$$T : C_c^\infty(\overline{\mathbb{R}}^n_+) \to C_c^\infty(\mathbb{R}^{n-1})$$

is therefore bounded with respect to the $W^{1,p}(\mathbb{R}^n_+)$ and $L^p(\partial\mathbb{R}^n_+)$ norms, and extends by density and continuity to a map between these spaces.

It follows immediately that $Tf = 0$ if $f \in W_0^{k,p}(\mathbb{R}^n_+)$. We omit the proof that $Tf = 0$ implies that $f \in W_0^{k,p}(\mathbb{R}^n_+)$ (see [**5**]). □

If $p = 1$, the trace $T : W^{1,1}(\mathbb{R}^n_+) \to Ł^1(\mathbb{R}^{n-1})$ is onto, but if $1 < p < \infty$ the range of T is not all of L^p. In that case, $T : W^{1,p}(\mathbb{R}^n_+) \to B^{1-1/p,p}(\mathbb{R}^{n-1})$ maps $W^{1,p}$ onto a Besov space $B^{1-1/p,p}$; roughly speaking, this is a Sobolev space of functions with fractional derivatives, and there is a loss of $1/p$ derivatives in restricting a function to the boundary [**14**].

Note that if $f \in W_0^{2,p}(\mathbb{R}_+^n)$, then $\partial_i f \in W_0^{1,p}(\mathbb{R}_+^n)$, so $T(\partial_i f) = 0$. Thus, both f and Df vanish on the boundary. The correct way to formulate the condition that f has weak derivatives of order less than or equal to two and satisfies the Dirichlet condition $f = 0$ on the boundary is that $f \in W^{2,p}(\mathbb{R}_+^n) \cap W_0^{1,p}(\mathbb{R}_+^n)$.

3.10. Compactness results

A Banach space X is compactly embedded in a Banach space Y, written $X \Subset Y$, if the embedding $\imath : X \to Y$ is compact. That is, $\imath$ maps bounded sets in X to precompact sets in Y; or, equivalently, if $\{x_n\}$ is a bounded sequence in X, then $\{\imath x_n\}$ has a convergent subsequence in Y.

An important property of the Sobolev embeddings is that they are compact on domains with finite measure. This corresponds to the rough principle that uniform bounds on higher derivatives imply compactness with respect to lower derivatives. The compactness of the Sobolev embeddings, due to Rellich and Kondrachov, depend on the Arzelà-Ascoli theorem. We will prove a version for $W_0^{1,p}(\Omega)$ by use of the L^p-compactness criterion in Theorem 1.14.

Theorem 3.43. *Let Ω be a bounded open set in $\mathbb{R}^n$, $1 \le p < n$, and $1 \le q < p^*$. If $\mathcal{F}$ is a bounded set in $W_0^{1,p}(\Omega)$, then $\mathcal{F}$ is precompact in $L^q(\mathbb{R}^n)$.*

Proof. By a density argument, we may assume that the functions in $\mathcal{F}$ are smooth and $\operatorname{spt} f \Subset \Omega$. We may then extend the functions and their derivatives by zero to obtain smooth functions on $\mathbb{R}^n$, and prove that $\mathcal{F}$ is precompact in $L^q(\mathbb{R}^n)$.

Condition (1) in Theorem 1.14 follows immediately from the boundedness of Ω and the Sobolev imbeddeding theorem: for all $f \in \mathcal{F}$,

$$\|f\|_{L^q(\mathbb{R}^n)} = \|f\|_{L^q(\Omega)} \le C\|f\|_{L^{p^*}(\Omega)} \le C\|Df\|_{L^p(\mathbb{R}^n)} \le C$$

where C denotes a generic constant that does not depend on f. Condition (2) is satisfied automatically since the supports of all functions in $\mathcal{F}$ are contained in the same bounded set.

To verify (3), we first note that since Df is supported inside the bounded open set Ω,

$$\|Df\|_{L^1(\mathbb{R}^n)} \le C\,\|Df\|_{L^p(\mathbb{R}^n)}\,.$$

Fix $h \in \mathbb{R}^n$ and let $f_h(x) = f(x+h)$ denote the translation of f by h. Then

$$|f_h(x) - f(x)| = \left|\int_0^1 h \cdot Df(x+th)\,dt\right| \le |h| \int_0^1 |Df(x+th)|\,dt.$$

Integrating this inequality with respect to x and using Fubini's theorem to exchange the order of integration on the right-hand side, together with the fact that the inner x-integral is independent of t, we get

$$\int_{\mathbb{R}^n} |f_h(x) - f(x)|\,dx \le |h|\,\|Df\|_{L^1(\mathbb{R}^n)} \le C|h|\,\|Df\|_{L^p(\mathbb{R}^n)}\,.$$

Thus,

$$\|f_h - f\|_{L^1(\mathbb{R}^n)} \le C|h|\,\|Df\|_{L^p(\mathbb{R}^n)}\,. \tag{3.15}$$

Using the interpolation inequality in Lemma 1.10, we get for any $1 \le q < p^*$ that

$$\|f_h - f\|_{L^q(\mathbb{R}^n)} \le \|f_h - f\|_{L^1(\mathbb{R}^n)}^{\theta}\,\|f_h - f\|_{L^{p^*}(\mathbb{R}^n)}^{1-\theta} \tag{3.16}$$

where $0 < \theta \leq 1$ is given by

$$\frac{1}{q} = \theta + \frac{1-\theta}{p^*}.$$

The Sobolev embedding theorem implies that

$$\|f_h - f\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Df\|_{L^p(\mathbb{R}^n)}.$$

Using this inequality and (3.15) in (3.16), we get

$$\|f_h - f\|_{L^q(\mathbb{R}^n)} \leq C|h|^\theta \|Df\|_{L^p(\mathbb{R}^n)}.$$

It follows that $\mathcal{F}$ is L^q-equicontinuous if the derivatives of functions in $\mathcal{F}$ are uniformly bounded in L^p, and the result follows. $\square$

Equivalently, this theorem states that if $\{f_k k \in \mathbb{N}\}$ is a sequence of functions in $W_0^{1,p}(\Omega)$ such that

$$\|f_k\|_{W^{1,p}} \leq C \qquad \text{for all } k \in \mathbb{N},$$

for some constant C, then there exists a subsequence f_{k_i} and a function $f \in L^q(\Omega)$ such that

$$f_{k_i} \to f \qquad \text{as } i \to \infty \text{ in } L^q(\Omega).$$

The assumptions that the domain Ω satisfies a boundedness condition and that $q < p^*$ are necessary.

Example 3.44. If $\phi \in W^{1,p}(\mathbb{R}^n)$ and $f_m(x) = \phi(x - c_m)$, where $c_m \to \infty$ as $m \to \infty$, then $\|f_m\|_{W^{1,p}} = \|\phi\|_{W^{1,p}}$ is constant, but $\{f_m\}$ has no convergent subsequence in L^q since the functions 'escape' to infinity. Thus, compactness does not hold without some limitation on the decay of the functions.

Example 3.45. For $1 \leq p < n$, define $f_k : \mathbb{R}^n \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} k^{n/p^*}(1 - k|x|) & \text{if } |x| < 1/k, \\ 0 & \text{if } |x| \geq 1/k. \end{cases}$$

Then $\operatorname{spt} f_k \subset \overline{B}_1(0)$ for every $k \in \mathbb{N}$ and $\{f_k\}$ is bounded in $W^{1,p}(\mathbb{R}^n)$, but no subsequence converges strongly in $L^{p^*}(\mathbb{R}^n)$.

The loss of compactness in the critical case $q = p^*$ has received a great deal of study (for example, in the concentration compactness principle of P.L. Lions).

If Ω is a smooth and bounded domain, the use of an extension map implies that $W^{1,p}(\Omega) \Subset L^q(\Omega)$. For an example of the loss of this compactness in a bounded domain with an irregular boundary, see [**14**].

Theorem 3.46. *Let Ω be a bounded open set in $\mathbb{R}^n$, and $n < p < \infty$. Suppose that $\mathcal{F}$ is a set of functions whose weak derivative belongs to $L^p(\mathbb{R}^n)$ such that: (a)* $\operatorname{spt} f \Subset \Omega$*; (b) there exists a constant C such that*

$$\|Df\|_{L^p} \leq C \qquad \text{for all } f \in \mathcal{F}.$$

Then $\mathcal{F}$ is precompact in $C_0(\mathbb{R}^n)$.

Proof. Theorem 3.34 implies that the set $\mathcal{F}$ is bounded and equicontinuous, so the result follows immediately from the Arzelà-Ascoli theorem. $\square$

In other words, if $\{f_m : m \in \mathbb{N}\}$ is a sequence of functions in $W^{1,p}(\mathbb{R}^n)$ such that $\operatorname{spt} f_m \subset \Omega$, where $\Omega \Subset \mathbb{R}^n$, and

$$\|f_m\|_{W^{1,p}} \le C \qquad \text{for all } m \in \mathbb{N}$$

for some constant C, then there exists a subsequence f_{m_k} such that $f_{n_k} \to f$ uniformly, in which case $f \in C_c(\mathbb{R}^n)$.

3.11. Sobolev functions on $\Omega \subset \mathbb{R}^n$

Here, we briefly outline how ones transfers the results above to Sobolev spaces on domains other than $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Suppose that Ω is a smooth, bounded domain in $\mathbb{R}^n$. We may cover the closure $\overline{\Omega}$ by a collection of open balls contained in Ω and open balls with center $x \in \partial\Omega$. Since $\overline{\Omega}$ is compact, there is a finite collection $\{B_i : 1 \le i \le N\}$ of such open balls that covers $\overline{\Omega}$. There is a partition of unity $\{\psi_i : 1 \le i \le N\}$ subordinate to this cover consisting of functions $\psi_i \in C_c^\infty(B_i)$ such that $0 \le \psi_i \le 1$ and $\sum_i \psi_i = 1$ on $\overline{\Omega}$.

Given any function $f \in L^1_{\text{loc}}(\Omega)$, we may write $f = \sum_i f_i$ where $f_i = \psi_i f$ has compact support in B_i for balls whose center belongs to Ω, and in $B_i \cap \overline{\Omega}$ for balls whose center belongs to $\partial\Omega$. In these latter balls, we may 'straighten out the boundary' by a smooth map. After this change of variables, we get a function f_i that is compactly supported in $\overline{\mathbb{R}}^n_+$. We may then apply the previous results to the functions $\{f_i : 1 \le i \le N\}$.

Typically, results about $W_0^{k,p}(\Omega)$ do not require assumptions on the smoothness of $\partial\Omega$; but results about $W^{k,p}(\Omega)$ — for example, the existence of a bounded extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ — only hold if $\partial\Omega$ satisfies an appropriate smoothness or regularity condition *e.g.* a C^k, Lipschitz, segment, or cone condition [**1**].

The statement of the imbedding theorem for higher order derivatives extends in a straightforward way from the one for first order derivatives. For example,

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \qquad \text{if} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

The result for smooth bounded domains is summarized in the following theorem. As before, $X \subset Y$ denotes a continuous imbedding of X into Y, and $X \Subset Y$ denotes a compact imbedding.

THEOREM 3.47. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^1 boundary, $k, m \in \mathbb{N}$ with $k \ge m$, and $1 \le p < \infty$.*

(1) *If $kp < n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \qquad \text{for } 1 \le q < np/(n-kp);$$
$$W^{k,p}(\Omega) \subset L^q(\Omega) \qquad \text{for } q = np/(n-kp).$$

More generally, if $(k-m)p < n$, then

$$W^{k,p}(\Omega) \Subset W^{m,q}(\Omega) \qquad \text{for } 1 \le q < np/\left(n-(k-m)p\right);$$
$$W^{k,p}(\Omega) \subset W^{m,q}(\Omega) \qquad \text{for } q = np/\left(n-(k-m)p\right).$$

(2) *If $kp = n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \qquad \text{for } 1 \le q < \infty.$$

(3) *If $kp > n$, then*

$$W^{k,p}(\Omega) \Subset C^{0,\mu}\left(\overline{\Omega}\right)$$

for $0 < \mu < k - n/p$ if $k - n/p < 1$, for $0 < \mu < 1$ if $k - n/p = 1$, and for $\mu = 1$ if $k - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{0,\mu}\left(\overline{\Omega}\right)$$

for $\mu = k - n/p$ if $k - n/p < 1$. More generally, if $(k - m)p > n$, then

$$W^{k,p}(\Omega) \Subset C^{m,\mu}\left(\overline{\Omega}\right)$$

for $0 < \mu < k-m-n/p$ if $k-m-n/p < 1$, for $0 < \mu < 1$ if $k-m-n/p = 1$, and for $\mu = 1$ if $k - m - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{m,\mu}\left(\overline{\Omega}\right)$$

for $\mu = k - m - n/p$ if $k - m - n/p = 0$.

These results hold for arbitrary bounded open sets Ω if $W^{k,p}(\Omega)$ is replaced by $W_0^{k,p}(\Omega)$.

Appendix

To understand weak derivatives and distributional derivatives in the simplest context of functions of a single variable, we describe without proof some results from real analysis (see [**7**] and [**8**]). These results are, in fact, easier to understand from the perspective of weak and distributional derivatives of functions, rather than pointwise derivatives. This discussion will not be needed below.

For definiteness, we consider functions $f : [a, b] \to \mathbb{R}$ defined on a compact interval $[a, b]$. When we say that a property holds almost everywhere (a.e.), we always mean a.e. with respect to Lebesgue measure unless we specify otherwise.

3.A. Lipschitz functions

Lipschitz continuity is a weaker condition than continuous differentiability. A Lipschitz continuous function is pointwise differentiable almost everwhere and weakly differentiable. The derivative is essentially bounded, but not necessarily continuous.

Definition 3.48. A function $f : [a, b] \to \mathbb{R}$ is uniformly Lipschitz continuous on $[a, b]$ (or Lipschitz, for short) if there is a constant C such that

$$|f(x) - f(y)| \leq C\,|x - y| \qquad \text{for all } x, y \in [a, b].$$

The Lipschitz constant of f is the infimum of constants C with this property.

We denote the space of Lipschitz functions on $[a, b]$ by $\mathrm{Lip}[a, b]$. We also define the space of locally Lipschitz functions on $\mathbb{R}$ by

$$\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \in \mathrm{Lip}[a, b] \text{ for all } a < b\}.$$

By the mean-value theorem, any function that is continuous on $[a, b]$ and pointwise differentiable in (a, b) with bounded derivative is Lipschitz. In particular, every function $f \in C^1([a, b])$ is Lipschitz, and every function $f \in C^1(\mathbb{R})$ is locally Lipschitz. On the other hand, the function $x \mapsto |x|$ is Lipschitz but not C^1 on $[-1, 1]$. The following result, called Rademacher's theorem, is true for functions of several variables, but we state it here only for the one-dimensional case.

Theorem 3.49. *If $f \in \mathrm{Lip}[a, b]$, then the pointwise derivative f' exists almost everywhere in (a, b) and is essentially bounded.*

It follows from the discussion in the next section that the pointwise derivative of a Lipschitz function is also its weak derivative (since a Lipschitz function is absolutely continuous). In fact, we have the following characterization of Lipschitz functions.

Theorem 3.50. *Suppose that $f \in L^1_{\mathrm{loc}}(a, b)$. Then $f \in \mathrm{Lip}[a, b]$ if and only if f is weakly differentiable in (a, b) and $f' \in L^\infty(a, b)$. Moreover, the Lipschitz constant of f is equal to the sup-norm of f'.*

Here, we say that $f \in L^1_{\mathrm{loc}}(a, b)$ is Lipschitz on $[a, b]$ if is equal almost everywhere to a (uniformly) Lipschitz function on (a, b), in which case f extends by uniform continuity to a Lipschitz function on $[a, b]$.

Example 3.51. The function $f(x) = x_+$ in Example 3.3 is Lipschitz continuous on $[-1, 1]$ with Lipschitz constant 1. The pointwise derivative of f exists everywhere except at $x = 0$, and is equal to the weak derivative. The sup-norm of the weak derivative $f' = \chi_{[0,1]}$ is equal to 1.

EXAMPLE 3.52. Consider the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right).$$

Since f is C^1 on compactly contained intervals in $(0,1)$, an integration by parts implies that

$$\int_0^1 f\phi'\,dx = -\int_0^1 f'\phi\,dx \qquad \text{for all } \phi \in C_c^\infty(0,1).$$

Thus, the weak derivative of f in $(0,1)$ is

$$f'(x) = -\cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right).$$

Since $f' \in L^\infty(0,1)$, f is Lipschitz on $[0,1]$,

Similarly, if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, if and only if f is weakly differentiable in $\mathbb{R}$ and $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

3.B. Absolutely continuous functions

Absolute continuity is a strengthening of uniform continuity that provides a necessary and sufficient condition for the fundamental theorem of calculus to hold. A function is absolutely continuous if and only if its weak derivative is integrable.

DEFINITION 3.53. A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^N |f(b_i) - f(a_i)| < \epsilon$$

for any finite collection $\{[a_i,b_i] : 1 \le i \le N\}$ of non-overlapping subintervals $[a_i,b_i]$ of $[a,b]$ with

$$\sum_{i=1}^N |b_i - a_i| < \delta$$

Here, we say that intervals are non-overlapping if their interiors are disjoint. We denote the space of absolutely continuous functions on $[a,b]$ by $\mathrm{AC}[a,b]$. We also define the space of locally absolutely continuous functions on $\mathbb{R}$ by

$$\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \in \mathrm{AC}[a,b] \text{ for all } a < b\}.$$

Restricting attention to the case $N = 1$ in Definition 3.53, we see that an absolutely continuous function is uniformly continuous, but the converse is not true (see Example 3.55).

EXAMPLE 3.54. A Lipschitz function is absolutely continuous. If the function has Lipschitz constant C, we may take $\delta = \epsilon/C$ in the definition of absolute continuity.

EXAMPLE 3.55. The Cantor function f in Example 3.5 is uniformly continuous on $[0,1]$, as is any continuous function on a compact interval, but it is not absolutely continuous. We may enclose the Cantor set in a union of disjoint intervals the sum of whose lengths is as small as we please, but the jumps in f across those intervals add up to 1. Thus for any $0 < \epsilon \le 1$, there is no $\delta > 0$ with the property required in the definition of absolute continuity. In fact, absolutely continuous functions map sets of measure zero to sets of measure zero; by contrast, the Cantor function maps the Cantor set with measure zero onto the interval $[0,1]$ with measure one.

EXAMPLE 3.56. If $g \in L^1(a,b)$ and

$$f(x) = \int_a^x g(t)\,dt$$

then $f \in \mathrm{AC}[a,b]$ and $f' = g$ pointwise a.e. (at every Lebesgue point of g). This is one direction of the fundamental theorem of calculus.

According to the following result, absolutely continuous functions are precisely the ones for which the fundamental theorem of calculus holds.

THEOREM 3.57. *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if: (a) the pointwise derivative f' exists almost everywhere in (a,b); (b) the derivative $f' \in L^1(a,b)$ is integrable; and (c) for every $x \in [a,b]$,*

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

To prove this result, one shows from the definition of absolute continuity that if $f \in \mathrm{AC}[a,b]$, then f' exists pointwise a.e. and is integrable, and if $f' = 0$, then f is constant. Then the function

$$f(x) - \int_a^x f'(t)\,dt$$

is absolutely continuous with pointwise a.e. derivative equal to zero, so the result follows.

EXAMPLE 3.58. We recover the function $f(x) = x_+$ in Example 3.3 by integrating its derivative $\chi_{[0,\infty)}$. On the other hand, the pointwise a.e. derivative of the Cantor function in Example 3.5 is zero, so integration of its pointwise derivative (which exists a.e. and is integrable) gives zero instead of the original function.

Integration by parts holds for absolutely continuous functions.

THEOREM 3.59. *If $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous, then*

$$\int_a^b fg'\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'g\,dx \tag{3.17}$$

where f', g' denote the pointwise a.e. derivatives of f, g.

This result is not true under the assumption that f, g that are continuous and differentiable pointwise a.e., as can be seen by taking f, g to be Cantor functions on $[0,1]$.

In particular, taking $g \in C_c^\infty(a,b)$ in (3.17), we see that an absolutely continuous function f is weakly differentiable on (a,b) with integrable derivative, and the weak derivative is equal to the pointwise a.e. derivative. Thus, we have the following characterization of absolutely continuous functions in terms of weak derivatives.

THEOREM 3.60. *Suppose that $f \in L^1_{\mathrm{loc}}(a,b)$. Then $f \in \mathrm{AC}[a,b]$ if and only if f is weakly differentiable in (a,b) and $f' \in L^1(a,b)$.*

It follows that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is weakly differentiable if and only if $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$, in which case $f' \in L^1_{\mathrm{loc}}(\mathbb{R})$.

3.C. Functions of bounded variation

Functions of bounded variation are functions with finite oscillation or variation. A function of bounded variation need not be weakly differentiable, but its distributional derivative is a Radon measure.

Definition 3.61. The total variation $\mathrm{V}_f([a,b])$ of a function $f : [a,b] \to \mathbb{R}$ on the interval $[a,b]$ is

$$\mathrm{V}_f([a,b]) = \sup\left\{\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|\right\}$$

where the supremum is taken over all partitions

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

of the interval $[a,b]$. A function f has bounded variation on $[a,b]$ if $\mathrm{V}_f([a,b])$ is finite.

We denote the space of functions of bounded variation on $[a,b]$ by $\mathrm{BV}[a,b]$, and refer to a function of bounded variation as a BV-function. We also define the space of locally BV-functions on $\mathbb{R}$ by

$$\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \in \mathrm{BV}[a,b] \text{ for all } a < b\}.$$

Example 3.62. Every Lipschitz continuous function $f : [a,b] \to \mathbb{R}$ has bounded variation, and

$$\mathrm{V}_f([a,b]) \le C(b-a)$$

where C is the Lipschitz constant of f.

A BV-function is bounded, and an absolutely continuous function is BV; but a BV-function need not be continuous, and a continuous function need not be BV.

Example 3.63. The discontinuous step function in Example 3.4 has bounded variation on the interval $[-1,1]$, and the continuous Cantor function in Example 3.5 has bounded variation on $[0,1]$. The total variation of both functions is equal to one. More generally, any monotone function $f : [a,b] \to \mathbb{R}$ has bounded variation, and its total variation on $[a,b]$ is equal to $|f(b) - f(a)|$.

Example 3.64. The function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is bounded $[0,1]$, but it is not of bounded variation on $[0,1]$.

Example 3.65. The function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous on $[0,1]$, but it is not of bounded variation on $[0,1]$ since its total variation is proportional to the divergent harmonic series $\sum 1/n$.

The following result states that any BV-functions is a difference of monotone increasing functions. We say that a function f is monotone increasing if $f(x) \le f(y)$ for $x \le y$; we do not require that the function is strictly increasing.

THEOREM 3.66. *A function $f : [a, b] \to \mathbb{R}$ has bounded variation on $[a, b]$ if and only if $f = f_+ - f_-$, where $f_+, f_- : [a, b] \to \mathbb{R}$ are bounded monotone increasing functions.*

To prove the theorem, we define an increasing variation function $v : [a, b] \to \mathbb{R}$ by $v(a) = 0$ and

$$v(x) = \mathrm{V}_f\left([a, x]\right) \qquad \text{for } x > a.$$

We then choose f_+, f_- so that

$$f = f_+ - f_-, \qquad v = f_+ + f_-, \tag{3.18}$$

and show that f_+, f_- are increasing functions.

The decomposition in Theorem 3.66 is not unique, since we may add an arbitrary increasing function to both f_+ and f_-, but it is unique if we add the condition that $f_+ + f_- = \mathrm{V}_f$.

A monotone function is differentiable pointwise a.e., and thus so is a BV-function. In general, a BV-function contains a singular component that is not weakly differentiable in addition to an absolutely continuous component that is weakly differentiable

DEFINITION 3.67. A function $f \in \mathrm{BV}[a, b]$ is singular on $[a, b]$ if the pointwise derivative f' is equal to zero a.e. in $[a, b]$.

The step function and the Cantor function are examples of non-constant singular functions.[3]

THEOREM 3.68. *If $f \in \mathrm{BV}[a, b]$, then $f = f_{ac} + f_s$ where $f_{ac} \in \mathrm{AC}[a, b]$ and f_s is singular. The functions f_{ac}, f_s are unique up to an additive constant.*

The absolutely continuous part f_{ac} of f is given by

$$f_{ac}(x) = \int_a^x f'(x)\, dx$$

and the remainder $f_s = f - f_{ac}$ is the singular part. We may further decompose the singular part into a jump-function (such as the step function) and a singular continuous part (such as the Cantor function).

For $f \in \mathrm{BV}[a, b]$, let $D \subset [a, b]$ denote the set of points of discontinuity of f. Since f is the difference of monotone functions, it can only contain jump discontinuities at which its left and right limits exist (excluding the left limit at a and the right limit at b), and D is necessarily countable.

If $c \in D$, let

$$[f](c) = f(c^+) - f(c^-)$$

denote the jump of f at c (with $f(a^-) = f(a)$, $f(b^+) = f(b)$ if $a, b \in D$). Define

$$f_p(x) = \sum_{c \in D \cap [a,x]} [f](c) \qquad \text{if } x \notin D.$$

Then f_p has the same jump discontinuities as f and, with an appropriate choice of $f_p(c)$ for $c \in D$, the function $f - f_p$ is continuous on $[a, b]$. Decomposing this continuous part into and absolutely continuous and a singular continuous part, we get the following result.

[3]Sometimes a singular function is required to be continuous, but our definition allows jump discontinuities.

THEOREM 3.69. *If $f \in \mathrm{BV}[a,b]$, then $f = f_{ac} + f_p + f_{sc}$ where $f_{ac} \in \mathrm{AC}[a,b]$, f_p is a jump function, and f_{sc} is a singular continuous function. The functions f_{ac}, f_p, f_{sc} are unique up to an additive constant.*

EXAMPLE 3.70. Let $Q = \{q_n : n \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0,1]$ and $\{p_n : n \in \mathbb{N}\}$ any sequence of real numbers such that $\sum p_n$ is absolutely convergent. Define $f : [a,b] \to \mathbb{R}$ by $f(0) = 0$ and

$$f(x) = \sum_{a \le q_n \le x} p_n \qquad \text{for } x > 0.$$

Then $f \in \mathrm{BV}[a,b]$, with

$$\mathrm{V}_f[a,b] = \sum_{n\in\mathbb{N}} |p_n|.$$

This function is a singular jump function with zero pointwise derivative at every irrational number in $[0,1]$.

3.D. Borel measures on $\mathbb{R}$

We denote the extended real numbers by $\overline{\mathbb{R}} = [-\infty, \infty]$ and the extended nonnegative real numbers by $\overline{\mathbb{R}}_+ = [0, \infty]$. We make the natural conventions for algebraic operations and limits that involve extended real numbers.

The Borel σ-algebra of a topological space X is the smallest collection of subsets of X that contains the open and closed sets, and is closed under complements, countable unions, and countable intersections. Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$, and $\overline{\mathcal{B}}$ the Borel σ-algebra of $\overline{\mathbb{R}}$.

DEFINITION 3.71. A Borel measure on $\mathbb{R}$ is a function $\mu : \mathcal{B} \to \overline{\mathbb{R}}_+$, such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

for any countable collection of disjoint sets $\{E_n \in \mathcal{B} : n \in \mathbb{N}\}$.

The measure μ is finite if $\mu(\mathbb{R}) < \infty$, in which case $\mu : \mathcal{B} \to [0, \infty)$. The measure is σ-finite if $\mathbb{R}$ is a countable union of Borel sets with finite measure.

EXAMPLE 3.72. Lebesgue measure $\lambda : \mathcal{B} \to \overline{\mathbb{R}}_+$ is a Borel measure that assigns to each interval its length. Lebesgue measure on $\mathcal{B}$ may be extended to a complete measure on a larger σ-algebra of Lebesgue measurable sets by the inclusion of all subsets of sets with Lebesgue measure zero. Here we consider it as a Borel measure.

EXAMPLE 3.73. For $c \in \mathbb{R}$, the unit point measure $\delta_c : \mathcal{B} \to [0, \infty)$ supported on c is defined by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

This measure is a finite Borel measure. More generally, if $\{c_n : n \in \mathbb{N}\}$ is a countable set of points in $\mathbb{R}$ and $\{p_n \ge 0 : n \in \mathbb{N}\}$, we define a point measure

$$\mu = \sum_{n\in\mathbb{N}} p_n \delta_{c_n}, \qquad \mu(E) = \sum_{c_n \in E} p_n.$$

This measure is σ-finite, and finite if $\sum p_n < \infty$.

EXAMPLE 3.74. Counting measure $\nu : \mathcal{B} \to \overline{\mathbb{R}}_+$ is defined by $\nu(E) = \#E$ where $\#E$ denotes the number of points in E. Thus, $\nu(\emptyset) = 0$ and $\nu(E) = \infty$ if E contains infinitely many points. This measure is not σ-finite.

In order to describe the decomposition of measures, we introduce the idea of singular measures that 'live' on different sets.

DEFINITION 3.75. Two measures $\mu, \nu : \mathcal{B} \to \overline{\mathbb{R}}_+$ are mutually singular, written $\mu \perp \nu$, if there is a set $E \in \mathcal{B}$ such that $\mu(E) = 0$ and $\nu(E^c) = 0$.

We also say that μ is singular with respect to ν, or ν is singular with respect to μ. In particular, a measure is singular with respect to Lebesgue measure if it assigns full measure to a set of Lebesgue measure zero.

EXAMPLE 3.76. The point measures in Example 3.73 are singular with respect to Lebesgue measure.

Next we consider signed measures which can take negative as well as positive values.

DEFINITION 3.77. A signed Borel measure is a map $\mu : \mathcal{B} \to \overline{\mathbb{R}}$ of the form

$$\mu = \mu_+ - \mu_-$$

where $\mu_+, \mu_- : \mathcal{B} \to \overline{\mathbb{R}}_+$ are Borel measures, at least one of which is finite.

The condition that at least one of μ_+, μ_- is finite is needed to avoid meaningless expressions such as $\mu(\mathbb{R}) = \infty - \infty$. Thus, μ takes at most one of the values ∞, $-\infty$.

According to the Jordan decomposition theorem, we may choose μ_+, μ_- in Definition 3.77 so that $\mu_+ \perp \mu_-$, in which case the decomposition is unique. The total variation of μ is then measure $|\mu| : \mathcal{B} \to \overline{\mathbb{R}}_+$ defined by

$$|\mu| = \mu_+ + \mu_-.$$

DEFINITION 3.78. Let $\mu : \mathcal{B} \to \overline{\mathbb{R}}_+$ be a measure. A signed measure $\nu : \mathcal{B} \to \overline{\mathbb{R}}$ is absolutely continuous with respect to μ, written $\nu \ll \mu$, if $\mu(E) = 0$ implies that $\nu(E) = 0$ for any $E \in \mathcal{B}$.

The condition $\nu \ll \mu$ is equivalent to $|\nu| \ll \mu$. In that case ν 'lives' on the same sets as μ; thus absolute continuity is at the opposite extreme to singularity. In particular, a signed measure ν is absolutely continuous with respect to Lebesgue measure if it assigns zero measure to any set with zero Lebesgue measure,

If $g \in L^1(\mathbb{R})$, then

$$\nu(E) = \int_E g\,dx \tag{3.19}$$

defines a finite signed Borel measure $\nu : \mathcal{B} \to \mathbb{R}$. This measure is absolutely continuous with respect to Lebesgue measure, since $\int_E g\,dx = 0$ for any set E with Lebesgue measure zero.

If $g \geq 0$, then ν is a measure. If the set $\{x : g(x) = 0\}$ has non-zero Lebesgue measure, then Lebesgue measure is not absolutely continuous with respect to ν. Thus $\nu \ll \mu$ does not imply that $\mu \ll \nu$.

The Radon-Nikodym theorem (which holds in greater generality) implies that every absolutely continuous measure is given by the above example.

THEOREM 3.79. *If ν is a Borel measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure λ then there exists a function $g \in L^1(\mathbb{R})$ such that ν is given by (3.19).*

The function g in this theorem is called the Radon-Nikodym derivative of ν with respect to λ, and is denoted by

$$g = \frac{d\nu}{d\lambda}.$$

The following result gives an alternative characterization of absolute continuity of measures, which has a direct connection with the absolute continuity of functions.

THEOREM 3.80. *A signed measure $\nu : \mathcal{B} \to \overline{\mathbb{R}}$ is absolutely continuous with respect to a measure $\mu : \mathcal{B} \to \overline{\mathbb{R}}_+$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies that $|\nu(E)| \le \epsilon$ for all $E \in \mathcal{B}$.*

3.E. Radon measures on $\mathbb{R}$

The most important Borel measures for distribution theory are the Radon measures. The essential property of a Radon measure μ is that integration against μ defines a positive linear functional on the space of continuous functions ϕ with compact support,

$$\phi \mapsto \int \phi \, d\mu.$$

(See Theorem 3.93 below.) This link is the fundamental connection between measures and distributions. The condition in the following definition characterizes all such measures on $\mathbb{R}$ (and $\mathbb{R}^n$).

DEFINITION 3.81. A Radon measure on $\mathbb{R}$ is a Borel measure that is finite on compact sets.

We note in passing that a Radon measure μ has the following regularity property: For any $E \in \mathcal{B}$,

$$\mu(E) = \inf \{\mu(G) : G \supset E \text{ open}\}, \quad \mu(E) = \sup \{\mu(K) : K \subset E \text{ compact}\}.$$

Thus, any Borel set may be approximated in a measure-theoretic sense by open sets from the outside and compact sets from the inside.

EXAMPLE 3.82. Lebesgue measure λ in Example 3.72 and the point measure δ_c in Example 3.73 are Radon measures on $\mathbb{R}$.

EXAMPLE 3.83. The counting measure ν in Example 3.74 is not a Radon measure since, for example, $\nu[0,1] = \infty$. This measure is not outer regular: If $\{c\}$ is a singleton set, then $\nu(\{c\}) = 1$ but

$$\inf \{\nu(G) : c \in G, \, G \text{ open}\} = \infty.$$

The following is the Lebesgue decomposition of a Radon measure.

THEOREM 3.84. *Let μ, ν be Radon measures on $\mathbb{R}$. There are unique measures ν_{ac}, ν_s such that*

$$\nu = \nu_{ac} + \nu_s, \qquad \text{where } \nu_{ac} \ll \mu \text{ and } \nu_s \perp \mu.$$

3.F. Lebesgue-Stieltjes measures

Given a Radon measure μ on $\mathbb{R}$, we may define a monotone increasing, right-continuous distribution function $f : \mathbb{R} \to \mathbb{R}$, which is unique up to an arbitrary additive constant, such that

$$\mu(a, b] = f(b) - f(a).$$

The function f is right-continuous since

$$\lim_{x \to b^+} f(b) - f(a) = \lim_{x \to b^+} \mu(a, x] = \mu(a, b] = f(b) - f(a).$$

Conversely, every such function f defines a Radon measure μ_f, called the Lebesgue-Stieltjes measure associated with f. Thus, Radon measures on $\mathbb{R}$ may be characterized explicitly as Lebesgue-Stieltjes measures.

Theorem 3.85. *If $f : \mathbb{R} \to \mathbb{R}$ is a monotone increasing, right-continuous function, there is a unique Radon measure μ_f such that*

$$\mu_f(a, b] = f(b) - f(a)$$

for any half-open interval $(a, b] \subset \mathbb{R}$.

The standard proof is due to Carathéodory. One uses f to define a countably sub-additive outer measure μ_f^* on all subsets of $\mathbb{R}$, then restricts μ_f^* to a measure on the σ-algebra of μ_f^*-measurable sets, which includes all of the Borel sets [**7**].

The Lebesgue-Stieltjes measure of a compact interval $[a, b]$ is given by

$$\mu_f[a, b] = \lim_{x \to a^-} \mu_f(x, b] = f(b) - \lim_{x \to a^-} f(a).$$

Thus, the measure of the set consisting of a single point is equal to the jump in f at the point,

$$\mu_f\{a\} = f(a) - \lim_{x \to a^-} f(a),$$

and $\mu_f\{a\} = 0$ if and only if f is continuous at a.

Example 3.86. If $f(x) = x$, then μ_f is Lebesgue measure (restricted to the Borel sets) in $\mathbb{R}$.

Example 3.87. If $c \in \mathbb{R}$ and

$$f(x) = \begin{cases} 1 & \text{if } x \geq c, \\ 0 & \text{if } x < c, \end{cases}$$

then μ_f is the point measure δ_c in Example 3.73.

Example 3.88. If f is the Cantor function defined in Example 3.5, then μ_f assigns measure one to the Cantor set C and measure zero to $\mathbb{R} \setminus C$. Thus, μ_f is singular with respect to Lebesgue measure. Nevertheless, since f is continuous, the measure of any set consisting of a single point, and therefore any countable set, is zero.

If $f : \mathbb{R} \to \mathbb{R}$ is the difference $f = f_+ - f_-$ of two right-continuous monotone increasing functions $f_+, f_- : \mathbb{R} \to \mathbb{R}$, at least one of which is bounded, we may define a signed Radon measure $\mu_f : \mathcal{B} \to \overline{\mathbb{R}}$ by

$$\mu_f = \mu_{f_+} - \mu_{f_-}.$$

If we add the condition that $\mu_{f_+} \perp \mu_{f_-}$, then this decomposition is unique, and corresponds to the decomposition of f in (3.18).

3.G. Integration

A function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable if $\phi^{-1}(E) \in \mathcal{B}$ for every $E \in \overline{\mathcal{B}}$. In particular, every continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Given a Borel measure μ, and a non-negative, Borel measurable function ϕ, we define the integral of ϕ with respect to μ as follows. If

$$\psi = \sum_{i \in \mathbb{N}} c_i \chi_{E_i}$$

is a simple function, where $c_i \in \overline{\mathbb{R}}_+$ and χ_{E_i} is the characteristic function of a set $E_i \in \mathcal{B}$, then

$$\int \psi \, d\mu = \sum_{i \in \mathbb{N}} c_i \mu(E_i).$$

Here, we define $0 \cdot \infty = 0$ for the integral of a zero value on a set of infinite measure, or an infinite value on a set of measure zero. If $\phi : \mathbb{R} \to \overline{\mathbb{R}}_+$ is a non-negative Borel-measurable function, we define

$$\int \phi \, d\mu = \sup \left\{ \int \psi \, d\mu : 0 \le \psi \le \phi \right\}$$

where the supremum is taken over all non-negative simple functions ψ that are bounded from above by ϕ.

If $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is a general Borel function, we split ϕ into its positive and negative parts,

$$\phi = \phi_+ - \phi_-, \qquad \phi_+ = \max(\phi, 0), \quad \phi_- = \max(-\phi, 0),$$

and define

$$\int \phi \, d\mu = \int \phi_+ \, d\mu - \int \phi_- \, d\mu$$

provided that at least one of these integrals is finite.[4]

Example 3.89. The integral of ϕ with respect to Lebesgue measure λ in Example 3.72 is the usual Lebesgue integral

$$\int \phi \, d\lambda = \int \phi \, dx.$$

Example 3.90. The integral of ϕ with respect to the point measure δ_c in Example 3.73 is

$$\int \phi \, d\delta_c = \phi(c).$$

Note that $\phi = \psi$ pointwise a.e. with respect to δ_c if and only if $\phi(c) = \psi(c)$.

Example 3.91. If f is absolutely continuous, the associated Lebesgue-Stieltjes measure μ_f is absolutely continuous with respect to Lebesgue measure, and

$$\int \phi \, d\mu_f = \int \phi f' \, dx.$$

[4]The continual annoyance of excluding $\infty - \infty$ as meaningless is often viewed as a defect of the Lebesgue integral, which cannot cope directly with the cancelation between infinite positive and negative components. For example, $\int_0^\infty \sin(x)/x \, dx = \pi/2$ is not true as a Lebesgue integral since $\int_0^\infty |\sin(x)/x| \, dx = \infty$. Nevertheless, other definitions of the integral have not proved to be as useful as the Lebesgue integral.

Next, we consider linear functionals on the space $C_c(\mathbb{R})$ of linear functions with compact support.

DEFINITION 3.92. A linear functional $I : C_c(\mathbb{R}) \to \mathbb{R}$ is positive if $I(\phi) \geq 0$ whenever $\phi \geq 0$, and locally bounded if for every compact set K in $\mathbb{R}$ there is a constant C_K such that

$$|I(\phi)| \leq C_K \|\phi\|_\infty \qquad \text{for all } \phi \in C_c(\mathbb{R}) \text{ with } \operatorname{spt}\phi \subset K.$$

A positive functional is locally bounded, and a locally bounded functional I defines a distribution $I \in \mathcal{D}'(\mathbb{R})$ by restriction to $C_c^\infty(\mathbb{R})$. We also write $I(\phi) = \langle I, \phi\rangle$. If μ is a Radon measure, then

$$\langle I_\mu, \phi\rangle = \int \phi\, d\mu$$

defines a positive linear functional $I_\mu : C_c(\mathbb{R}) \to \mathbb{R}$, and if μ_+, μ_- are Radon measures, then $I_{\mu_+} - I_{\mu_-}$ is a locally bounded functional.

Conversely, according to the Riesz representation theorem, all locally bounded linear functionals on $C_c(\mathbb{R})$ are of this form

THEOREM 3.93. *If $I : C_c(\mathbb{R}) \to \mathbb{R}_+$ is a positive linear functional on the space of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ with compact support, then there is a unique Radon measure μ such that*

$$I(\phi) = \int \phi\, d\mu.$$

If $I : C_c(\mathbb{R}) \to \overline{\mathbb{R}}_+$ is locally bounded linear functional, then there are unique Radon measures μ_+, μ_- such that

$$I(\phi) = \int \phi\, d\mu_+ - \int \phi\, d\mu_-.$$

Note that $\mu = \mu_+ - \mu_-$ is not well-defined as a signed Radon measure if both μ_+ and μ_- are infinite.

Every distribution $T \in \mathcal{D}'(\mathbb{R})$ such that

$$\langle T, \phi\rangle \leq C_K \|\phi\|_\infty \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}) \text{ with } \operatorname{spt}\phi \subset K$$

may be extended by continuity to a locally bounded linear functional on $C_c(\mathbb{R})$, and therefore is given by $T = I_{\mu_+} - I_{\mu_-}$ for Radon measures μ_+, μ_- . We typically identify a Radon measure μ with the corresponding distribution I_μ. If μ is absolutely continuous with respect to Lebesgue measure, then $\mu = \mu_f$ for some $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$ and I_μ is the same as the regular distribution $T_{f'}$. Thus, with these identifications, we have the following inclusions:

$$\mathrm{AC} \subset \mathrm{BV} \subset \text{Integrable functions} \subset \text{Radon measures} \subset \text{Distributions}.$$

The distributional derivative of an AC function is an integrable function, and the following integration by parts formula shows that the distributional derivative of a BV function is a Radon measure.

THEOREM 3.94. *Suppose that $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ and $g \in \mathrm{AC}_c(\mathbb{R})$ is absolutely continuous with compact support. Then*

$$\int g\, d\mu_f = -\int f g'\, dx.$$

Thus, the distributional derivative of $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ is the functional I_{μ_f} associated with the corresponding Radon measure μ_f. If

$$f = f_{ac} + f_p + f_{sc}$$

is the decomposition of f into a locally absolutely continuous part, a jump function, and a singular continuous function, then

$$\mu_f = \mu_{ac} + \mu_p + \mu_{sc},$$

where μ_{ac} is absolutely continuous with respect to Lebesgue measure with density f'_{ac}, μ_p is a point measure of the form

$$\mu_p = \sum_{n\in\mathbb{N}} p_n \delta_{c_n}$$

where the c_n are the points of discontinuity of f and the p_n are the jumps, and μ_{sc} is a measure with continuous distribution function that is singular with respect to Lebesgue measure. The function is weakly differentiable if and only if it is locally absolutely continuous.

Thus, to return to our original one-dimensional examples, the function x_+ in Example 3.3 is absolutely continuous and its weak derivative is the step function. The weak derivative is bounded since the function is Lipschitz. The step function in Example 3.4 is not weakly differentiable; its distributional derivative is the δ-measure. The Cantor function f in Example 3.5 is not weakly differentiable; its distributional derivative is the singular continuous Lebesgue-Stieltjes measure μ_f associated with f.

3.H. Summary

We summarize the above discussion in a table.

Function	*Weak Derivative*
Smooth (C^1)	Continuous (C^0)
Lipschitz	Bounded (L^∞)
Absolutely Continuous	Integrable (L^1)
Bounded Variation	Distributional derivative is Radon measure

The correspondences shown in this table continue to hold for functions of several variables, although the study of fine structure of weakly differentiable functions and functions of bounded variation is more involved than in the one-dimensional case.

CHAPTER 4

Elliptic PDEs

One of the main advantages of extending the class of solutions of a PDE from classical solutions with continuous derivatives to weak solutions with weak derivatives is that it is easier to prove the existence of weak solutions. Having established the existence of weak solutions, one may then study their properties, such as uniqueness and regularity, and perhaps prove under appropriate assumptions that the weak solutions are, in fact, classical solutions.

There is often considerable freedom in how one defines a weak solution of a PDE; for example, the function space to which a solution is required to belong is not given *a priori* by the PDE itself. Typically, we look for a weak formulation that reduces to the classical formulation under appropriate smoothness assumptions and which is amenable to a mathematical analysis; the notion of solution and the spaces to which solutions belong are dictated by the available estimates and analysis.

4.1. Weak formulation of the Dirichlet problem

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain Ω in $\mathbb{R}^n$,

$$-\Delta u = f \qquad \text{in } \Omega, \tag{4.1}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4.2}$$

First, suppose that the boundary of Ω is smooth and $u, f : \overline{\Omega} \to \mathbb{R}$ are smooth functions. Multiplying (4.1) by a test function ϕ, integrating the result over Ω, and using the divergence theorem, we get

$$\int_\Omega Du \cdot D\phi \, dx = \int_\Omega f\phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{4.3}$$

The boundary terms vanish because $\phi = 0$ on the boundary. Conversely, if f and Ω are smooth, then any smooth function u that satisfies (4.3) is a solution of (4.1).

Next, we formulate weaker assumptions under which (4.3) makes sense. We use the flexibility of choice to define weak solutions with L^2-derivatives that belong to a Hilbert space; this is helpful because Hilbert spaces are easier to work with than Banach spaces.[1] It also leads to a variational form of the equation that is symmetric in the solution u and the test function ϕ.

By the Cauchy-Schwartz inequality, the integral on the left-hand side of (4.3) is finite if Du belongs to $L^2(\Omega)$, so we suppose that $u \in H^1(\Omega)$. We impose the boundary condition (4.2) in a weak sense by requiring that $u \in H_0^1(\Omega)$. The left hand side of (4.3) then extends by continuity to $\phi \in H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}$.

[1]We would need to use Banach spaces to study the solutions of Laplace's equation whose derivatives lie in L^p for $p \neq 2$, and we may be forced to use Banach spaces for some PDEs, especially if they are nonlinear.

The right hand side of (4.3) is well-defined for all $\phi \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$, but this is not the most general f for which it makes sense; we can define the right-hand for any f in the dual space of $H_0^1(\Omega)$.

DEFINITION 4.1. The space of bounded linear maps $f : H_0^1(\Omega) \to \mathbb{R}$ is denoted by $H^{-1}(\Omega) = H_0^1(\Omega)^*$, and the action of $f \in H^{-1}(\Omega)$ on $\phi \in H_0^1(\Omega)$ by $\langle f, \phi\rangle$. The norm of $f \in H^{-1}(\Omega)$ is given by

$$\|f\|_{H^{-1}} = \sup\left\{\frac{|\langle f, \phi\rangle|}{\|\phi\|_{H_0^1}} : \phi \in H_0^1, \phi \neq 0\right\}.$$

A function $f \in L^2(\Omega)$ defines a linear functional $F_f \in H^{-1}(\Omega)$ by

$$\langle F_f, v\rangle = \int_\Omega f v\, dx = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega).$$

Here, $(\cdot,\cdot)_{L^2}$ denotes the standard inner product on $L^2(\Omega)$. The functional F_f is bounded on $H_0^1(\Omega)$ with $\|F_f\|_{H^{-1}} \leq \|f\|_{L^2}$ since, by the Cauchy-Schwartz inequality,

$$|\langle F_f, v\rangle| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H_0^1}.$$

We identify F_f with f, and write both simply as f.

Such linear functionals are, however, not the only elements of $H^{-1}(\Omega)$. As we will show below, $H^{-1}(\Omega)$ may be identified with the space of distributions on Ω that are sums of first-order distributional derivatives of functions in $L^2(\Omega)$.

Thus, after identifying functions with regular distributions, we have the following triple of Hilbert spaces

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \qquad H^{-1}(\Omega) = H_0^1(\Omega)^*.$$

Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, then

$$\langle f, u\rangle = (f, u)_{L^2},$$

so the duality pairing coincides with the L^2-inner product when both are defined.

This discussion motivates the following definition.

DEFINITION 4.2. Let Ω be an open set in $\mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is a weak solution of (4.1)–(4.2) if: (a) $u \in H_0^1(\Omega)$; (b)

$$\int_\Omega Du \cdot D\phi\, dx = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega). \tag{4.4}$$

Here, strictly speaking, 'function' means an equivalence class of functions with respect to pointwise a.e. equality.

We have assumed homogeneous boundary conditions to simplify the discussion. If Ω is smooth and $g : \partial\Omega \to \mathbb{R}$ is a function on the boundary that is in the range of the trace map $T : H^1(\Omega) \to L^2(\partial\Omega)$, say $g = Tw$, then we obtain a weak formulation of the nonhomogeneous Dirichet problem

$$\begin{aligned} -\Delta u &= f &&\text{in } \Omega,\\ u &= g &&\text{on } \partial\Omega, \end{aligned}$$

by replacing (a) in Definition 4.2 with the condition that $u - w \in H_0^1(\Omega)$. The definition is otherwise the same. The range of the trace map on $H^1(\Omega)$ for a smooth domain Ω is the fractional-order Sobolev space $H^{1/2}(\partial\Omega)$; thus if the boundary data g is so rough that $g \notin H^{1/2}(\partial\Omega)$, then there is no solution $u \in H^1(\Omega)$ of the nonhomogeneous BVP.

4.2. Variational formulation

Definition 4.2 of a weak solution in is closely connected with the variational formulation of the Dirichlet problem for Poisson's equation. To explain this connection, we first summarize some definitions of the differentiability of functionals (scalar-valued functions) acting on a Banach space.

DEFINITION 4.3. A functional $J : X \to \mathbb{R}$ on a Banach space X is differentiable at $x \in X$ if there is a bounded linear functional $A : X \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|J(x+h) - J(x) - Ah|}{\|h\|_X} = 0.$$

If A exists, then it is unique, and it is called the derivative, or differential, of J at x, denoted $DJ(x) = A$.

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If J is differentiable at every point of X, then $DJ : X \to X^*$ maps $x \in X$ to the linear functional $DJ(x) \in X^*$ that approximates J near x.

A weaker notion of differentiability (even for functions $J : \mathbb{R}^2 \to \mathbb{R}$ — see Example 4.4) is the existence of directional derivatives

$$\delta J(x;h) = \lim_{\epsilon\to 0}\left[\frac{J(x+\epsilon h) - J(x)}{\epsilon}\right] = \frac{d}{d\epsilon} J(x+\epsilon h)\bigg|_{\epsilon=0}.$$

If the directional derivative at x exists for every $h \in X$ and is a bounded linear functional on h, then $\delta J(x;h) = \delta J(x)h$ where $\delta J(x) \in X^*$. We call $\delta J(x)$ the Gâteaux derivative of J at x. The derivative DJ is then called the Fréchet derivative to distinguish it from the directional or Gâteaux derivative. If J is differentiable at x, then it is Gâteaux-differentiable at x and $DJ(x) = \delta J(x)$, but the converse is not true.

EXAMPLE 4.4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x,y) = \left(\frac{xy^2}{x^2+y^4}\right)^2 \qquad \text{if } (x,y) \neq (0,0).$$

Then f is Gâteaux-differentiable at 0, with $\delta f(0) = 0$, but f is not Fréchet-differentiable at 0.

If $J : X \to \mathbb{R}$ attains a local minimum at $x \in X$ and J is differentiable at x, then for every $h \in X$ the function $J_{x;h} : \mathbb{R} \to \mathbb{R}$ defined by $J_{x;h}(t) = J(x+th)$ is differentiable at $t = 0$ and attains a minimum at $t = 0$. It follows that

$$\frac{dJ_{x;h}}{dt}(0) = \delta J(x;h) = 0 \qquad \text{for every } h \in X.$$

Hence $DJ(x) = 0$. Thus, just as in multivariable calculus, an extreme point of a differentiable functional is a critical point where the derivative is zero.

Given $f \in H^{-1}(\Omega)$, define a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx - \langle f, u\rangle. \tag{4.5}$$

Clearly, J is well-defined.

PROPOSITION 4.5. *The functional $J : H_0^1(\Omega) \to \mathbb{R}$ in (4.5) is differentiable. Its derivative $DJ(u) : H_0^1(\Omega) \to \mathbb{R}$ at $u \in H_0^1(\Omega)$ is given by*

$$DJ(u)h = \int_\Omega Du \cdot Dh\, dx - \langle f, h\rangle \qquad \text{for } h \in H_0^1(\Omega).$$

PROOF. Given $u \in H_0^1(\Omega)$, define the linear map $A : H_0^1(\Omega) \to \mathbb{R}$ by

$$Ah = \int_\Omega Du \cdot Dh\, dx - \langle f, h\rangle.$$

Then A is bounded, with $\|A\| \leq \|Du\|_{L^2} + \|f\|_{H^{-1}}$, since

$$|Ah| \leq \|Du\|_{L^2}\|Dh\|_{L^2} + \|f\|_{H^{-1}}\|h\|_{H_0^1} \leq \left(\|Du\|_{L^2} + \|f\|_{H^{-1}}\right)\|h\|_{H_0^1}.$$

For $h \in H_0^1(\Omega)$, we have

$$J(u+h) - J(u) - Ah = \frac{1}{2}\int_\Omega |Dh|^2\, dx.$$

It follows that

$$|J(u+h) - J(u) - Ah| \leq \frac{1}{2}\|h\|_{H_0^1}^2,$$

and therefore

$$\lim_{h\to 0} \frac{|J(u+h) - J(u) - Ah|}{\|h\|_{H_0^1}} = 0,$$

which proves that J is differentiable on $H_0^1(\Omega)$ with $DJ(u) = A$. □

Note that $DJ(u) = 0$ if and only if u is a weak solution of Poisson's equation in the sense of Definition 4.2. Thus, we have the following result.

COROLLARY 4.6. *If $J : H_0^1(\Omega) \to \mathbb{R}$ defined in (4.5) attains a minimum at $u \in H_0^1(\Omega)$, then u is a weak solution of $-\Delta u = f$ in the sense of Definition 4.2.*

In the direct method of the calculus of variations, we prove the existence of a minimizer of J by showing that a minimizing sequence $\{u_n\}$ converges in a suitable sense to a minimizer u. This minimizer is then a weak solution of (4.1)–(4.2). We will not follow this method here, and instead establish the existence of a weak solution by use of the Riesz representation theorem. The Riesz representation theorem is, however, typically proved by a similar argument to the one used in the direct method of the calculus of variations, so in essence the proofs are equivalent.

4.3. The space $H^{-1}(\Omega)$

The negative order Sobolev space $H^{-1}(\Omega)$ can be described as a space of distributions on Ω.

THEOREM 4.7. *The space $H^{-1}(\Omega)$ consists of all distributions $f \in \mathcal{D}'(\Omega)$ of the form*

$$f = f_0 + \sum_{i=1}^n \partial_i f_i \qquad \text{where } f_0, f_i \in L^2(\Omega). \tag{4.6}$$

These distributions extend uniquely by continuity from $\mathcal{D}(\Omega)$ to bounded linear functionals on $H_0^1(\Omega)$. Moreover,

$$\|f\|_{H^{-1}(\Omega)} = \inf\left\{\left(\sum_{i=0}^n \int_\Omega f_i^2\, dx\right)^{1/2} : \text{such that } f_0,\ f_i \text{ satisfy (4.6)}\right\}. \tag{4.7}$$

PROOF. First suppose that $f \in H^{-1}(\Omega)$. By the Riesz representation theorem there is a function $g \in H_0^1(\Omega)$ such that

$$\langle f, \phi\rangle = (g, \phi)_{H_0^1} \qquad \text{for all } \phi \in H_0^1(\Omega). \tag{4.8}$$

Here, $(\cdot,\cdot)_{H_0^1}$ denotes the standard inner product on $H_0^1(\Omega)$,

$$(u, v)_{H_0^1} = \int_\Omega (uv + Du \cdot Dv)\, dx.$$

Identifying a function $g \in L^2(\Omega)$ with its corresponding regular distribution, restricting f to $\phi \in \mathcal{D}(\Omega) \subset H_0^1(\Omega)$, and using the definition of the distributional derivative, we have

$$\begin{aligned}
\langle f, \phi\rangle &= \int_\Omega g\phi\, dx + \sum_{i=1}^n \int_\Omega \partial_i g\, \partial_i \phi\, dx \\
&= \langle g, \phi\rangle + \sum_{i=1}^n \langle \partial_i g, \partial_i \phi\rangle \\
&= \left\langle g - \sum_{i=1}^n \partial_i g_i, \phi \right\rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega),
\end{aligned}$$

where $g_i = \partial_i g \in L^2(\Omega)$. Thus the restriction of every $f \in H^{-1}(\Omega)$ from $H_0^1(\Omega)$ to $\mathcal{D}(\Omega)$ is a distribution

$$f = g - \sum_{i=1}^n \partial_i g_i$$

of the form (4.6). Also note that taking $\phi = g$ in (4.8), we get $\langle f, g\rangle = \|g\|_{H_0^1}^2$, which implies that

$$\|f\|_{H^{-1}} \geq \|g\|_{H_0^1} = \left(\int_\Omega g^2\, dx + \sum_{i=1}^n \int_\Omega g_i^2\, dx \right)^{1/2},$$

which proves inequality in one direction of (4.7).

Conversely, suppose that $f \in \mathcal{D}'(\Omega)$ is a distribution of the form (4.6). Then, using the definition of the distributional derivative, we have for any $\phi \in \mathcal{D}(\Omega)$ that

$$\langle f, \phi\rangle = \langle f_0, \phi\rangle + \sum_{i=1}^n \langle \partial_i f_i, \phi\rangle = \langle f_0, \phi\rangle - \sum_{i=1}^n \langle f_i, \partial_i \phi\rangle.$$

Use of the Cauchy-Schwartz inequality gives

$$|\langle f, \phi\rangle| \leq \left(\langle f_0, \phi\rangle^2 + \sum_{i=1}^n \langle f_i, \partial_i \phi\rangle^2 \right)^{1/2}.$$

Moreover, since the f_i are regular distributions belonging to $L^2(\Omega)$

$$|\langle f_i, \partial_i \phi\rangle| = \left| \int_\Omega f_i \partial_i \phi\, dx \right| \leq \left(\int_\Omega f_i^2\, dx \right)^{1/2} \left(\int_\Omega \partial_i \phi^2\, dx \right)^{1/2},$$

so

$$|\langle f, \phi\rangle| \leq \left[\left(\int_\Omega f_0^2\, dx \right) \left(\int_\Omega \phi^2\, dx \right) + \sum_{i=1}^n \left(\int_\Omega f_i^2\, dx \right) \left(\int_\Omega \partial_i \phi^2\, dx \right) \right]^{1/2},$$

and

$$
\begin{aligned}
|\langle f, \phi \rangle| &\leq \left(\int_\Omega f_0^2 \, dx + \sum_{i=1}^n \int_\Omega f_i^2 \, dx \right)^{1/2} \left(\int_\Omega \phi^2 + \int_\Omega \partial_i \phi^2 \, dx \right)^{1/2} \\
&\leq \left(\sum_{i=0}^n \int_\Omega f_i^2 \, dx \right)^{1/2} \|\phi\|_{H_0^1}
\end{aligned}
$$

Thus the distribution $f : \mathcal{D}(\Omega) \to \mathbb{R}$ is bounded with respect to the $H_0^1(\Omega)$-norm on the dense subset $\mathcal{D}(\Omega)$. It therefore extends in a unique way to a bounded linear functional on $H_0^1(\Omega)$, which we still denote by f. Moreover,

$$
\|f\|_{H^{-1}} \leq \left(\sum_{i=0}^n \int_\Omega f_i^2 \, dx \right)^{1/2},
$$

which proves inequality in the other direction of (4.7). □

The dual space of $H^1(\Omega)$ cannot be identified with a space of distributions on Ω because $\mathcal{D}(\Omega)$ is not a dense subspace. Any linear functional $f \in H^1(\Omega)^*$ defines a distribution by restriction to $\mathcal{D}(\Omega)$, but the same distribution arises from different linear functionals. Conversely, any distribution $T \in \mathcal{D}'(\Omega)$ that is bounded with respect to the H^1-norm extends uniquely to a bounded linear functional on H_0^1, but the extension of the functional to the orthogonal complement $(H_0^1)^\perp$ in H^1 is arbitrary (subject to maintaining its boundedness). Roughly speaking, distributions are defined on functions whose boundary values or trace is zero, but general linear functionals on H^1 depend on the trace of the function on the boundary $\partial\Omega$.

Example 4.8. The one-dimensional Sobolev space $H^1(0,1)$ is imbedded in the space $C([0,1])$ of continuous functions, since $p > n$ for $p = 2$ and $n = 1$. In fact, according to the Sobolev imbedding theorem $H^1(0,1) \hookrightarrow C^{0,1/2}([0,1])$, as can be seen directly from the Cauchy-Schwartz inequality:

$$
\begin{aligned}
|f(x) - f(y)| &\leq \int_y^x |f'(t)| \, dt \\
&\leq \left(\int_y^x 1 \, dt \right)^{1/2} \left(\int_y^x |f'(t)|^2 \, dt \right)^{1/2} \\
&\leq \left(\int_0^1 |f'(t)|^2 \, dt \right)^{1/2} |x - y|^{1/2}.
\end{aligned}
$$

As usual, we identify an element of $H^1(0,1)$ with its continuous representative in $C([0,1])$. By the trace theorem,

$$
H_0^1(0,1) = \left\{ u \in H^1(0,1) : u(0) = u(1) \right\}.
$$

The orthogonal complement is

$$
H_0^1(0,1)^\perp = \left\{ u \in H^1(0,1) : \text{such that } (u,v)_{H^1} = 0 \text{ for every } v \in H_0^1(0,1) \right\}.
$$

This condition implies that $u \in H_0^1(0,1)^\perp$ if and only if

$$
\int_0^1 (uv + u'v') \, dx = 0 \qquad \text{for all } v \in H_0^1(0,1),
$$

which means that u is a weak solution of the ODE

$$-u'' + u = 0.$$

It follows that $u(x) = c_1 e^x + c_2 e^{-x}$, so

$$H^1(0,1) = H_0^1(0,1) \oplus E$$

where E is the two dimensional subspace of $H^1(0,1)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$. Thus,

$$H^1(0,1)^* = H^{-1}(0,1) \oplus E^*.$$

If $f \in H^1(0,1)^*$ and $u = u_0 + c_1 e^x + c_2 e^{-x}$ where $u_0 \in H_0^1(0,1)$, then

$$\langle f, u\rangle = \langle f_0, u_0\rangle + a_1 c_1 + a_2 c_2$$

where $f_0 \in H^{-1}(0,1)$ is the restriction of f to $H_0^1(0,1)$ and

$$a_1 = \langle f, e^x\rangle, \qquad a_2 = \langle f, e^{-x}\rangle.$$

The constants a_1, a_2 determine how the functional $f \in H^1(0,1)^*$ acts on the boundary values $u(0)$, $u(1)$ of a function $u \in H^1(0,1)$.

4.4. The Poincaré inequality for $H_0^1(\Omega)$

We cannot, in general, estimate a norm of a function in terms of a norm of its derivative since constant functions have zero derivative. Such estimates are possible if we add an additional condition that eliminates non-zero constant functions. For example, we can require that the function vanishes on the boundary of a domain, or that it has zero mean. We typically also need some sort of boundedness condition on the domain of the function, since even if a function vanishes at some point we cannot expect to estimate the size of a function over arbitrarily large distances by the size of its derivative. The resulting inequalities are called Poincaré inequalities.

The inequality we prove here is a basic example of a Poincaré inequality. We say that an open set Ω in $\mathbb{R}^n$ is bounded in some direction if there is a unit vector $e \in \mathbb{R}^n$ and constants a, b such that $a < x \cdot e < b$ for all $x \in \Omega$.

THEOREM 4.9. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded is some direction. Then there is a constant C such that*

$$\int_\Omega u^2\, dx \leq C \int_\Omega |Du|^2\, dx \qquad \text{for all } u \in H_0^1(\Omega). \tag{4.9}$$

PROOF. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove the inequality for $u \in C_c^\infty(\Omega)$. The inequality is invariant under rotations and translations, so we can assume without loss of generality that the domain is bounded in the x_n-direction and lies between $0 < x_n < a$.

Writing $x = (x', x_n)$ where $x' = (x_1, \dots, , x_{n-1})$, we have

$$|u(x', x_n)| = \left|\int_0^{x_n} \partial_n u(x', t)\, dt\right| \leq \int_0^a |\partial_n u(x', t)|\, dt.$$

The Cauchy-Schwartz inequality implies that

$$\int_0^a |\partial_n u(x', t)|\, dt = \int_0^a 1 \cdot |\partial_n u(x', t)|\, dt \leq a^{1/2} \left(\int_0^a |\partial_n u(x', t)|^2\, dt\right)^{1/2}.$$

Hence,

$$|u(x', x_n)|^2 \leq a \int_0^a |\partial_n u(x', t)|^2 \, dt.$$

Integrating this inequality with respect to x_n, we get

$$\int_0^a |u(x', x_n)|^2 \, dx_n \leq a^2 \int_0^a |\partial_n u(x', t)|^2 \, dt.$$

A further integration with respect to x' gives

$$\int_\Omega |u(x)|^2 \, dx \leq a^2 \int_\Omega |\partial_n u(x)|^2 \, dx.$$

Since $|\partial_n u| \leq |Du|$, the result follows with $C = a^2$. □

This inequality implies that we may use as an equivalent inner-product on H_0^1 an expression that involves only the derivatives of the functions and not the functions themselves.

COROLLARY 4.10. *If Ω is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product*

$$(u, v)_0 = \int_\Omega Du \cdot Dv \, dx \tag{4.10}$$

is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

PROOF. We denote the norm associated with the inner-product (4.10) by

$$\|u\|_0 = \left(\int_\Omega |Du|^2 \, dx \right)^{1/2},$$

and the standard norm and inner product by

$$\begin{aligned} \|u\|_1 &= \left(\int_\Omega \left[u^2 + |Du|^2 \right] dx \right)^{1/2}, \\ (u, v)_1 &= \int_\Omega (uv + Du \cdot Dv) \, dx. \end{aligned} \tag{4.11}$$

Then, using the Poincaré inequality (4.9), we have

$$\|u\|_0 \leq \|u\|_1 \leq (C + 1)^{1/2} \|u\|_0.$$

Thus, the two norms are equivalent; in particular, $(H_0^1, (\cdot, \cdot)_0)$ is complete since $(H_0^1, (\cdot, \cdot)_1)$ is complete, so it is a Hilbert space with respect to the inner product (4.10). □

4.5. Existence of weak solutions of the Dirichlet problem

With these preparations, the existence of weak solutions is an immediate consequence of the Riesz representation theorem.

THEOREM 4.11. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in the sense of Definition 4.2.*

PROOF. We equip $H_0^1(\Omega)$ with the inner product (4.10). Then, since Ω is bounded in some direction, the resulting norm is equivalent to the standard norm, and f is a bounded linear functional on $\left(H_0^1(\Omega), (,)_0\right)$. By the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, \phi)_0 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega),$$

which is equivalent to the condition that u is a weak solution. □

The same approach works for other symmetric linear elliptic PDEs. Let us give some examples.

EXAMPLE 4.12. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Then $u \in H_0^1(\Omega)$ is a weak solution if

$$\int_\Omega (Du \cdot D\phi + u\phi)\, dx = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

where $(\cdot, \cdot)_1$ is the standard inner product on $H_0^1(\Omega)$ given in (4.11). Thus, the Riesz representation theorem implies the existence of a unique weak solution.

Note that in this example and the next, we do not use the Poincaré inequality, so the result applies to arbitrary open sets, including $\Omega = \mathbb{R}^n$. In that case, $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, and we get a unique solution $u \in H^1(\mathbb{R}^n)$ of $-\Delta u + u = f$ for every $f \in H^{-1}(\mathbb{R}^n)$. Moreover, using the standard norms, we have $\|u\|_{H^1} = \|f\|_{H^{-1}}$. Thus the operator $-\Delta + I$ is an isometry of $H^1(\mathbb{R}^n)$ onto $H^{-1}(\mathbb{R}^n)$.

EXAMPLE 4.13. As a slight generalization of the previous example, suppose that $\mu > 0$. A function $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{aligned} -\Delta u + \mu u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.12}$$

if $(u, \phi)_\mu = \langle f, \phi \rangle$ for all $\phi \in H_0^1(\Omega)$ where

$$(u, v)_\mu = \int_\Omega (\mu uv + Du \cdot Dv)\, dx$$

The norm $\|\cdot\|_\mu$ associated with this inner product is equivalent to the standard one, since

$$\frac{1}{C}\|u\|_\mu^2 \leq \|u\|_1^2 \leq C\|u\|_\mu^2$$

where $C = \max\{\mu, 1/\mu\}$. We therefore again get the existence of a unique weak solution from the Riesz representation theorem.

EXAMPLE 4.14. Consider the last example for $\mu < 0$. If we have a Poincaré inequality $\|u\|_{L^2} \leq C\|Du\|_L^2$ for Ω, which is the case if Ω is bounded in some direction, then

$$(u, u)_\mu = \int_\Omega \left(\mu u^2 + Du \cdot Dv\right)\, dx \geq (1 - C|\mu|) \int_\Omega |Du|^2\, dx.$$

Thus $\|u\|_\mu$ defines a norm on $H_0^1(\Omega)$ that is equivalent to the standard norm if $-1/C < \mu < 0$, and we get a unique weak solution in this case also.

For bounded domains, the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ such that there exists a nonzero solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda_n u$. The best constant in the Poincaré inequality can be shown to be the minimum eigenvalue λ_1, and this method does not work if $\mu \leq -\lambda_1$. For $\mu = -\lambda_n$, a weak solution of (4.12) does not exist for every $f \in H^{-1}(\Omega)$, and if one does exist it is not unique since we can add to it an arbitrary eigenfunction. Thus, not only does the method fail, but the conclusion of Theorem 4.11 may be false.

Example 4.15. Consider the second order PDE

$$
\begin{aligned}
-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
\tag{4.13}
$$

where the coefficient functions $a_{ij} : \Omega \to \mathbb{R}$ are symmetric ($a_{ij} = a_{ji}$), bounded, and satisfy the uniform ellipticity condition that for some $\theta > 0$

$$
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n.
$$

Also, assume that Ω is bounded in some direction. Then a weak formulation of (4.13) is that $u \in H_0^1(\Omega)$ and

$$
a(u, \phi) = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega),
$$

where the symmetric bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$
a(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_i u \partial_j v \, dx.
$$

The boundedness of a_{ij}, the uniform ellipticity condition, and the Poincaré inequality imply that a defines an inner product on H_0^1 which is equivalent to the standard one. An application of the Riesz representation theorem for the bounded linear functionals f on the Hilbert space (H_0^1, a) then implies the existence of a unique weak solution. We discuss a generalization of this example in greater detail in the next section.

4.6. General linear, second order elliptic PDEs

Consider PDEs of the form

$$
Lu = f
$$

where L is a linear differential operator of the form

$$
Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu,
\tag{4.14}
$$

acting on functions $u : \Omega \to \mathbb{R}$ where Ω is an open set in $\mathbb{R}^n$. A physical interpretation of such PDEs is described briefly in Section 4.A.

We assume that the given coefficients functions $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ satisfy

$$
a_{ij}, b_i, c \in L^\infty(\Omega), \qquad a_{ij} = a_{ji}.
\tag{4.15}
$$

The operator L is elliptic if the matrix (a_{ij}) is positive definite. We will assume the stronger condition of uniformly ellipticity given in the next definition.

DEFINITION 4.16. The operator L in (4.14) is uniformly elliptic on Ω if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{4.16}$$

for x almost everywhere in Ω and every $\xi \in \mathbb{R}^n$.

This uniform ellipticity condition allows us to estimate the integral of $|Du|^2$ in terms of the integral of $\sum a_{ij}\partial_i u \partial_j u$.

EXAMPLE 4.17. The Laplacian operator $L = -\Delta$ is uniformly elliptic on any open set, with $\theta = 1$.

EXAMPLE 4.18. The Tricomi operator

$$L = y\partial_x^2 + \partial_y^2$$

is elliptic in $y > 0$ and hyperbolic in $y < 0$. For any $0 < \epsilon < 1$, L is uniformly elliptic in the strip $\{(x, y) : \epsilon < y < 1\}$, with $\theta = \epsilon$, but it is not uniformly elliptic in $\{(x, y) : 0 < y < 1\}$.

For $\mu \in \mathbb{R}$, we consider the Dirichlet problem for $L + \mu I$,

$$\begin{aligned} Lu + \mu u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.17}$$

We motivate the definition of a weak solution of (4.17) in a similar way to the motivation for the Laplacian: multiply the PDE by a test function $\phi \in C_c^\infty(\Omega)$, integrate over Ω, and use integration by parts, assuming that all functions and the domain are smooth. Note that

$$\int_\Omega \partial_i(b_i u)\phi\, dx = -\int_\Omega b_i u \partial_i \phi\, dx.$$

This leads to the condition that $u \in H_0^1(\Omega)$ is a weak solution of (4.17) with L given by (4.14) if

$$\int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j \phi - \sum_{i=1}^{n} b_i u \partial_i \phi + cu\phi \right\} dx + \mu \int_\Omega u\phi\, dx = \langle f, \phi \rangle$$

for all $\phi \in H_0^1(\Omega)$.

To write this condition more concisely, we define a bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j v - \sum_{i}^{n} b_i u \partial_i v + cuv \right\} dx. \tag{4.18}$$

This form is well-defined and bounded on $H_0^1(\Omega)$, as we check explicitly below. We denote the L^2-inner product by

$$(u, v)_{L^2} = \int_\Omega uv\, dx.$$

DEFINITION 4.19. Suppose that Ω is an open set in $\mathbb{R}^n$, $f \in H^{-1}(\Omega)$, and L is a differential operator (4.14) whose coefficients satisfy (4.15). Then $u : \Omega \to \mathbb{R}$ is a weak solution of (4.17) if: (a) $u \in H_0^1(\Omega)$; (b)

$$a(u, \phi) + \mu(u, \phi)_{L^2} = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

The form a in (4.18) is not symmetric unless $b_i = 0$. We have

$$a(v, u) = a^*(u, v)$$

where

$$a^*(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v + \sum_i^n b_i(\partial_i u)v + cuv \right\} dx \tag{4.19}$$

is the bilinear form associated with the formal adjoint L^* of L,

$$L^* u = -\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j u\right) - \sum_{i=1}^n b_i \partial_i u + cu. \tag{4.20}$$

The proof of the existence of a weak solution of (4.17) is similar to the proof for the Dirichlet Laplacian, with one exception. If L is not symmetric, we cannot use a to define an equivalent inner product on $H_0^1(\Omega)$ and appeal to the Riesz representation theorem. Instead we use a result due to Lax and Milgram which applies to non-symmetric bilinear forms.[2]

4.7. The Lax-Milgram theorem and general elliptic PDEs

We begin by stating the Lax-Milgram theorem for a bilinear form on a Hilbert space. Afterwards, we verify its hypotheses for the bilinear form associated with a general second-order uniformly elliptic PDE and use it to prove the existence of weak solutions.

THEOREM 4.20. *Let $\mathcal{H}$ be a Hilbert space with inner-product $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, and let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form on $\mathcal{H}$. Assume that there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|u\|^2 \le a(u, u), \quad |a(u, v)| \le C_2\|u\|\,\|v\| \qquad \text{for all } u, v \in \mathcal{H}.$$

Then for every bounded linear functional $f : \mathcal{H} \to \mathbb{R}$, there exists a unique $u \in \mathcal{H}$ such that

$$\langle f, v\rangle = a(u, v) \qquad \text{for all } v \in \mathcal{H}.$$

For the proof, see [**5**]. The verification of the hypotheses for (4.18) depends on the following energy estimates.

---

[2]The story behind this result — the story might be completely true or completely false — is that Lax and Milgram attended a seminar where the speaker proved existence for a symmetric PDE by use of the Riesz representation theorem, and one of them asked the other if symmetry was required; in half an hour, they convinced themselves that is wasn't, giving birth to the Lax-Milgram "lemma."

THEOREM 4.21. *Let a be the bilinear form on $H_0^1(\Omega)$ defined in (4.18), where the coefficients satisfy (4.15) and the uniform ellipticity condition (4.16) with constant θ. Then there exist constants $C_1, C_2 > 0$ and $\gamma \in \mathbb{R}$ such that for all $u, v \in H_0^1(\Omega)$*

$$C_1 \|u\|_{H_0^1}^2 \leq a(u,u) + \gamma \|u\|_{L^2}^2 \tag{4.21}$$

$$|a(u,v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1}, \tag{4.22}$$

If $b = 0$, we may take $\gamma = \theta - c_0$ where $c_0 = \inf_\Omega c$, and if $b \neq 0$, we may take

$$\gamma = \frac{1}{2\theta} \sum_{i=1}^n \|b_i\|_{L^\infty}^2 + \frac{\theta}{2} - c_0.$$

PROOF. First, we have for any $u, v \in H_0^1(\Omega)$ that

$$\begin{aligned}
|a(u,v)| &\leq \sum_{i,j=1}^n \int_\Omega |a_{ij} \partial_i u \partial_j v| \, dx + \sum_{i=1}^n \int_\Omega |b_i u \partial_i v| \, dx + \int_\Omega |cuv| \, dx. \\
&\leq \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} \|\partial_i u\|_{L^2} \|\partial_j v\|_{L^2} \\
&\quad + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i v\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\
&\leq C \left(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} + \sum_{i=1}^n \|b_i\|_{L^\infty} + \|c\|_{L^\infty} \right) \|u\|_{H_0^1} \|v\|_{H_0^1},
\end{aligned}$$

which shows (4.22).

Second, using the uniform ellipticity condition (4.16), we have

$$\begin{aligned}
\theta \|Du\|_{L^2}^2 &= \theta \int_\Omega |Du|^2 \, dx \\
&\leq \sum_{i,j=1}^n \int_\Omega a_{ij} \partial_i u \partial_j u \, dx \\
&\leq a(u,u) + \sum_{i=1}^n \int_\Omega b_i u \partial_i u \, dx - \int_\Omega c u^2 \, dx \\
&\leq a(u,u) + \sum_{i=1}^n \int_\Omega |b_i u \partial_i u| \, dx - c_0 \int_\Omega u^2 \, dx \\
&\leq a(u,u) + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i u\|_{L^2} - c_0 \|u\|_{L^2} \\
&\leq a(u,u) + \beta \|u\|_{L^2} \|Du\|_{L^2} - c_0 \|u\|_{L^2},
\end{aligned}$$

where $c(x) \geq c_0$ a.e. in Ω, and

$$\beta = \left(\sum_{i=1}^n \|b_i\|_{L^\infty}^2 \right)^{1/2}.$$

If $\beta = 0$, we get (4.21) with

$$\gamma = \theta - c_0, \qquad C_1 = \theta.$$

If $\beta > 0$, by Cauchy's inequality with ϵ, we have for any $\epsilon > 0$ that

$$\|u\|_{L^2} \|Du\|_{L^2} \leq \epsilon \|Du\|_{L^2}^2 + \frac{1}{4\epsilon} \|u\|_{L^2}^2 .$$

Hence, choosing $\epsilon = \theta/2\beta$, we get

$$\frac{\theta}{2}\|Du\|_{L^2}^2 \leq a(u,u) + \left(\frac{\beta^2}{2\theta} - c_0\right) \|u\|_{L^2} ,$$

and (4.21) follows with

$$\gamma = \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0, \qquad C_1 = \frac{\theta}{2}.$$

□

Equation (4.21) is called Gårding's inequality; this estimate of the H_0^1-norm of u in terms of $a(u,u)$, using the uniform ellipticity of L, is the crucial energy estimate. Equation (4.22) states that the bilinear form a is bounded on H_0^1. The expression for γ in this Theorem is not necessarily sharp. For example, as in the case of the Laplacian, the use of Poincaré's inequality gives smaller values of γ for bounded domains.

THEOREM 4.22. *Suppose that Ω is an open set in $\mathbb{R}^n$, and $f \in H^{-1}(\Omega)$. Let L be a differential operator (4.14) with coefficients that satisfy (4.15), and let $\gamma \in \mathbb{R}$ be a constant for which Theorem 4.21 holds. Then for every $\mu \geq \gamma$ there is a unique weak solution of the Dirichlet problem*

$$Lu + \mu f = 0, \qquad u \in H_0^1(\Omega)$$

in the sense of Definition 4.19.

PROOF. For $\mu \in \mathbb{R}$, define $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a_\mu(u,v) = a(u,v) + \mu(u,v)_{L^2} \tag{4.23}$$

where a is defined in (4.18). Then $u \in H_0^1(\Omega)$ is a weak solution of $Lu + \mu u = f$ if and only if

$$a_\mu(u,\phi) = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

From (4.22),

$$|a_\mu(u,v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1} + |\mu| \|u\|_{L^2}\|v\|_{L^2} \leq (C_2 + |\mu|) \|u\|_{H_0^1} \|v\|_{H_0^1}$$

so a_μ is bounded on $H_0^1(\Omega)$. From (4.21),

$$C_1\|u\|_{H_0^1}^2 \leq a(u,u) + \gamma\|u\|_{L^2}^2 \leq a_\mu(u,u)$$

whenever $\mu \geq \gamma$. Thus, by the Lax-Milgram theorem, for every $f \in H^{-1}(\Omega)$ there is a unique $u \in H_0^1(\Omega)$ such that $\langle f, \phi\rangle = a_\mu(u,\phi)$ for all $v \in H_0^1(\Omega)$, which proves the result. □

Although L^* is not of exactly the same form as L, since it first derivative term is not in divergence form, the same proof of the existence of weak solutions for L applies to L^* with a in (4.18) replaced by a^* in (4.19).

4.8. Compactness of the resolvent

An elliptic operator $L + \mu I$ of the type studied above is a bounded, invertible linear map from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ for sufficiently large $\mu \in \mathbb{R}$, so we may define an inverse operator $K = (L + \mu I)^{-1}$. If Ω is a bounded open set, then the Sobolev imbedding theorem implies that $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, and therefore K is a compact operator on $L^2(\Omega)$.

The operator $(L - \lambda I)^{-1}$ is called the resolvent of L, so this property is sometimes expressed by saying that L has compact resolvent. As discussed in Example 4.14, $L + \mu I$ may fail to be invertible at smaller values of μ, such that $\lambda = -\mu$ belongs to the spectrum $\sigma(L)$ of L, and the resolvent is not defined as a bounded operator on $L^2(\Omega)$ for $\lambda \in \sigma(L)$.

The compactness of the resolvent of elliptic operators on bounded open sets has several important consequences for the solvability of the elliptic PDE and the spectrum of the elliptic operator. Before describing some of these, we discuss the resolvent in more detail.

From Theorem 4.22, for $\mu \geq \gamma$ we can define

$$K : L^2(\Omega) \to L^2(\Omega), \qquad K = (L + \mu I)^{-1}\big|_{L^2(\Omega)}.$$

We define the inverse K on $L^2(\Omega)$, rather than $H^{-1}(\Omega)$, in which case its range is a subspace of $H_0^1(\Omega)$. If the domain Ω is sufficiently smooth for elliptic regularity theory to apply, then $u \in H^2(\Omega)$ if $f \in L^2(\Omega)$, and the range of K is $H^2(\Omega) \cap H_0^1(\Omega)$; for non-smooth domains, the range of K is more difficult to describe.

If we consider L as an operator acting in $L^2(\Omega)$, then the domain of L is $D = \operatorname{ran} K$, and

$$L : D \subset L^2(\Omega) \to L^2(\Omega)$$

is an unbounded linear operator with dense domain D. The operator L is closed, meaning that if $\{u_n\}$ is a sequence of functions in D such that $u_n \to u$ and $Lu_n \to f$ in $L^2(\Omega)$, then $u \in D$ and $Lu = f$. By using the resolvent, we can replace an analysis of the unbounded operator L by an analysis of the bounded operator K.

If $f \in L^2(\Omega)$, then $\langle f, v \rangle = (f, v)_{L^2}$. It follows from the definition of weak solution of $Lu + \mu u = f$ that

$$Kf = u \quad \text{if and only if} \quad a_\mu(u, v) = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega) \tag{4.24}$$

where a_μ is defined in (4.23). We also define the operator

$$K^* : L^2(\Omega) \to L^2(\Omega), \qquad K^* = (L^* + \mu I)^{-1}\big|_{L^2(\Omega)},$$

meaning that

$$K^* f = u \quad \text{if and only if} \quad a_\mu^*(u, v) = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega) \tag{4.25}$$

where $a_\mu^*(u, v) = a^*(u, v) + \mu\, (u, v)_{L^2}$ and a^* is given in (4.19).

THEOREM 4.23. *If $K \in \mathcal{B}\left(L^2(\Omega)\right)$ is defined by (4.24), then the adjoint of K is K^* defined by (4.25). If Ω is a bounded open set, then K is a compact operator.*

PROOF. If $f, g \in L^2(\Omega)$ and $Kf = u$, $K^* g = v$, then using (4.24) and (4.25), we get

$$(f, K^* g)_{L^2} = (f, v)_{L^2} = a_\mu(u, v) = a_\mu^*(v, u) = (g, u)_{L^2} = (u, g)_{L^2} = (Kf, g)_{L^2}.$$

Hence, K^* is the adjoint of K.

If $Kf = u$, then (4.21) with $\mu \geq \gamma$ and (4.24) imply that

$$C_1 \|u\|_{H_0^1}^2 \leq a_\mu(u,u) = (f,u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H_0^1}.$$

Hence $\|Kf\|_{H_0^1} \leq C\|f\|_{L^2}$ where $C = 1/C_1$. It follows that K is compact if Ω is bounded, since it maps bounded sets in $L^2(\Omega)$ into bounded sets in $H_0^1(\Omega)$, which are precompact in $L^2(\Omega)$ by the Sobolev imbedding theorem. □

4.9. The Fredholm alternative

Consider the Dirichlet problem

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.26}$$

where Ω is a smooth, bounded open set, and

$$Lu = -\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^n \partial_i \left(b_i u\right) + cu.$$

If $u = v = 0$ on $\partial\Omega$, Green's formula implies that

$$\int_\Omega (Lu)v \, dx = \int_\Omega u \left(L^* v\right) \, dx,$$

where the formal adjoint L^* of L is defined by

$$L^* v = -\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j v\right) - \sum_{i=1}^n b_i \partial_i v + cv.$$

It follows that if u is a smooth solution of (4.26) and v is a smooth solution of the homogeneous adjoint problem,

$$L^* v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then

$$\int_\Omega fv \, dx = \int_\Omega (Lu)v \, dx = \int_\Omega uL^* v \, dx = 0.$$

Thus, a necessary condition for (4.26) to be solvable is that f is orthogonal with respect to the $L^2(\Omega)$-inner product to every solution of the homogeneous adjoint problem.

For bounded domains, we will use the compactness of the resolvent to prove that this condition is necessary and sufficient for the existence of a weak solution of (4.26) where $f \in L^2(\Omega)$. Moreover, the solution is unique if and only if a solution exists for every $f \in L^2(\Omega)$.

This result is a consequence of the fact that if K is compact, then the operator $I + \sigma K$ is a Fredholm operator with index zero on $L^2(\Omega)$ for any $\sigma \in \mathbb{R}$, and therefore satisfies the Fredholm alternative (see Section 4.B.2). Thus, if $K = (L + \mu I)^{-1}$ is compact, the inverse elliptic operator $L - \lambda I$ also satisfies the Fredholm alternative.

THEOREM 4.24. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and L is a uniformly elliptic operator of the form (4.14) whose coefficients satisfy (4.15). Let L^* be the adjoint operator (4.20) and $\lambda \in \mathbb{R}$. Then one of the following two alternatives holds.*

1. *The only weak solution of the equation $L^* v - \lambda v = 0$ is $v = 0$. For every $f \in L^2(\Omega)$ there is a unique weak solution $u \in H_0^1(\Omega)$ of the equation $Lu - \lambda u = f$. In particular, the only solution of $Lu - \lambda u = 0$ is $u = 0$.*

(2) *The equation $L^*v - \lambda v = 0$ has a nonzero weak solution v. The solution spaces of $Lu - \lambda u = 0$ and $L^*v - \lambda v = 0$ are finite-dimensional and have the same dimension. For $f \in L^2(\Omega)$, the equation $Lu - \lambda u = f$ has a weak solution $u \in H_0^1(\Omega)$ if and only if $(f, v) = 0$ for every $v \in H_0^1(\Omega)$ such that $L^*v - \lambda v = 0$, and if a solution exists it is not unique.*

PROOF. Since $K = (L + \mu I)^{-1}$ is a compact operator on $L^2(\Omega)$, the Fredholm alternative holds for the equation

$$u + \sigma K u = g \qquad u, g \in L^2(\Omega) \tag{4.27}$$

for any $\sigma \in \mathbb{R}$. Let us consider the two alternatives separately.

First, suppose that the only solution of $v + \sigma K^* v = 0$ is $v = 0$, which implies that the only solution of $L^*v + (\mu + \sigma)v = 0$ is $v = 0$. Then the Fredholm alterative for $I + \sigma K$ implies that (4.27) has a unique solution $u \in L^2(\Omega)$ for every $g \in L^2(\Omega)$. In particular, for any $g \in \operatorname{ran} K$, there exists a unique solution $u \in L^2(\Omega)$, and the equation implies that $u \in \operatorname{ran} K$. Hence, we may apply $L + \mu I$ to (4.27), and conclude that for every $f = (L + \mu I)g \in L^2(\Omega)$, there is a unique solution $u \in \operatorname{ran} K \subset H_0^1(\Omega)$ of the equation

$$Lu + (\mu + \sigma)u = f. \tag{4.28}$$

Taking $\sigma = -(\lambda + \mu)$, we get part (1) of the Fredholm alternative for L.

Second, suppose that $v + \sigma K^* v = 0$ has a finite-dimensional subspace of solutions $v \in L^2(\Omega)$. It follows that $v \in \operatorname{ran} K^*$ (clearly, $\sigma \neq 0$ in this case) and

$$L^*v + (\mu + \sigma)v = 0.$$

By the Fredholm alternative, the equation $u + \sigma K u = 0$ has a finite-dimensional subspace of solutions of the same dimension, and hence so does

$$Lu + (\mu + \sigma)u = 0.$$

Equation (4.27) is solvable for $u \in L^2(\Omega)$ given $g \in \operatorname{ran} K$ if and only if

$$(v, g)_{L^2} = 0 \qquad \text{for all } v \in L^2(\Omega) \text{ such that } v + \sigma K^* v = 0, \tag{4.29}$$

and then $u \in \operatorname{ran} K$. It follows that the condition (4.29) with $g = Kf$ is necessary and sufficient for the solvability of (4.28) given $f \in L^2(\Omega)$. Since

$$(v, g)_{L^2} = (v, Kf)_{L^2} = (K^* v, f)_{L^2} = -\frac{1}{\sigma}(v, f)_{L^2}$$

and $v + \sigma K^* v = 0$ if and only if $L^*v + (\mu + \sigma)v = 0$, we conclude that (4.28) is solvable for u if and only if $f \in L^2(\Omega)$ satisfies

$$(v, f)_{L^2} = 0 \qquad \text{for all } v \in \operatorname{ran} K \text{ such that } L^*v + (\mu + \sigma)v = 0.$$

Taking $\sigma = -(\lambda + \mu)$, we get alternative (2) for L. □

Elliptic operators on a Riemannian manifold may have nonzero Fredholm index. The Atiyah-Singer index theorem (1968) relates the Fredholm index of such operators with a topological index of the manifold.

4.10. The spectrum of a self-adjoint elliptic operator

Suppose that L is a symmetric, uniformly elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + cu \tag{4.30}$$

where $a_{ij} = a_{ji}$ and $a_{ij}, c \in L^\infty(\Omega)$. The associated symmetric bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

is given by

$$a(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j u + cuv \right) dx.$$

The resolvent $K = (L + \mu I)^{-1}$ is a compact self-adjoint operator on $L^2(\Omega)$ for sufficiently large μ. Therefore its eigenvalues are real and its eigenfunctions provide an orthonormal basis of $L^2(\Omega)$. Since L has the same eigenfunctions as K, we get the corresponding result for L.

Theorem 4.25. *The operator L has an increasing sequence of real eigenvalues of finite multiplicity*

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \le \ldots$$

such that $\lambda_n \to \infty$. There is an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $L^2(\Omega)$ consisting of eigenfunctions functions $\phi_n \in H_0^1(\Omega)$ such that

$$L\phi_n = \lambda_n \phi_n.$$

Proof. If $K\phi = 0$ for any $\phi \in L^2(\Omega)$, then applying $L + \mu I$ to the equation we find that $\phi = 0$, so 0 is not an eigenvalue of K. If $K\phi = \kappa\phi$, for $\phi \in L^2(\Omega)$ and $\kappa \neq 0$, then $\phi \in \operatorname{ran} K$ and

$$L\phi = \left(\frac{1}{\kappa} - \mu\right)\phi,$$

so ϕ is an eigenfunction of L with eigenvalue $\lambda = 1/\kappa - \mu$. From Gårding's inequality (4.21) with $u = \phi$, and the fact that $a(\phi, \phi) = \lambda\|\phi\|_{L^2}^2$, we get

$$C_1\|\phi\|_{H_0^1}^2 \le (\lambda + \gamma)\|\phi\|_{L^2}^2.$$

It follows that $\lambda > -\gamma$, so the eigenvalues of L are bounded from below, and at most a finite number are negative. The spectral theorem for the compact self-adjoint operator K then implies the result. □

The boundedness of the domain Ω is essential here, otherwise K need not be compact, and the spectrum of L need not consist only of eigenvalues.

Example 4.26. Suppose that $\Omega = \mathbb{R}^n$ and $L = -\Delta$. Let $K = (-\Delta + I)^{-1}$. Then, from Example 4.12, $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The range of K is $H^2(\mathbb{R}^n)$. This operator is bounded but not compact. For example, if $\phi \in C_c^\infty(\mathbb{R}^n)$ is any nonzero function and $\{a_j\}$ is a sequence in $\mathbb{R}^n$ such that $|a_j| \uparrow \infty$ as $j \to \infty$, then the sequence $\{\phi_j\}$ defined by $\phi_j(x) = \phi(x - a_j)$ is bounded in $L^2(\mathbb{R}^n)$ but $\{K\phi_j\}$ has no convergent subsequence. In this example, K has continuous spectrum $[0, 1]$ on $L^2(\mathbb{R}^n)$ and no eigenvalues. Correspondingly, $-\Delta$ has the purely continuous spectrum $[0, \infty)$.

Finally, let us briefly consider the Fredholm alternative for a self-adjoint elliptic equation from the perspective of this spectral theory. The equation

$$Lu - \lambda u = f \tag{4.31}$$

may be solved by expansion with respect to the eigenfunctions of L. Suppose that $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$ such that $L\phi_n = \lambda_n \phi_n$, where the eigenvalues λ_n are increasing and repeated according to their multiplicity. We get the following alternatives, where all series converge in $L^2(\Omega)$:

(1) If $\lambda \neq \lambda_n$ for any $n \in \mathbb{N}$, then (4.31) has the unique solution

$$u = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n$$

for every $f \in L^2(\Omega)$;

(2) If $\lambda = \lambda_M$ for for some $M \in \mathbb{N}$ and $\lambda_n = \lambda_M$ for $M \leq n \leq N$, then (4.31) has a solution $u \in H_0^1(\Omega)$ if and only if $f \in L^2(\Omega)$ satisfies

$$(f, \phi_n) = 0 \qquad \text{for } M \leq n \leq N.$$

In that case, the solutions are

$$u = \sum_{\lambda_n \neq \lambda} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n + \sum_{n=M}^{N} c_n \phi_n$$

where $\{c_M, \ldots, c_N\}$ are arbitrary real constants.

4.11. Interior regularity

Roughly speaking, solutions of elliptic PDEs are as smooth as the data allows. For boundary value problems, it is convenient to consider the regularity of the solution in the interior of the domain and near the boundary separately. We begin by studying the interior regularity of solutions. We follow closely the presentation in [**5**].

To motivate the regularity theory, consider the following simple *a priori* estimate for the Laplacian. Suppose that $u \in C_c^\infty(\mathbb{R}^n)$. Then, integrating by parts twice, we get

$$\begin{aligned}
\int (\Delta u)^2 \, dx &= \sum_{i,j=1}^{n} \int \left(\partial_{ii}^2 u\right) \left(\partial_{jj}^2 u\right) \, dx \\
&= -\sum_{i,j=1}^{n} \int \left(\partial_{iij}^3 u\right) \left(\partial_j u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int \left(\partial_{ij}^2 u\right) \left(\partial_{ij}^2 u\right) \, dx \\
&= \int \left|D^2 u\right|^2 \, dx.
\end{aligned}$$

Hence, if $-\Delta u = f$, then

$$\left\|D^2 u\right\|_{L^2} = \|f\|_{L^2}^2.$$

Thus, we can control the L^2-norm of all second derivatives of u by the L^2-norm of the Laplacian of u. This estimate suggests that we should have $u \in H^2_{\text{loc}}$ if $f, u \in L^2$, as is in fact true. The above computation is, however, not justified for weak solutions that belong to H^1; as far as we know from the previous existence theory, such solutions may not even possess second-order weak derivatives.

We will consider a PDE

$$Lu = f \qquad \text{in } \Omega \tag{4.32}$$

where Ω is an open set in $\mathbb{R}^n$, $f \in L^2(\Omega)$, and L is a uniformly elliptic of the form

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right). \tag{4.33}$$

It is straightforward to extend the proof of the regularity theorem to uniformly elliptic operators that contain lower-order terms [**5**].

A function $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33) if

$$a(u, v) = (f, v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{4.34}$$

where the bilinear form a is given by

$$a(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_i u \partial_j v \, dx. \tag{4.35}$$

We do not impose any boundary condition on u, for example by requiring that $u \in H_0^1(\Omega)$, so the interior regularity theorem applies to any weak solution of (4.32).

Before stating the theorem, we illustrate the idea of the proof with a further *a priori* estimate. To obtain a local estimate for D^2u on a subdomain $\Omega' \Subset \Omega$, we introduce a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω'. We take as a test function

$$v = -\partial_k \eta^2 \partial_k u. \tag{4.36}$$

Note that v is given by a positive-definite, symmetric operator acting on u of a similar form to L, which leads to the positivity of the resulting estimate for $D\partial_k u$.

Multiplying (4.32) by v and integrating over Ω, we get $(Lu, v) = (f, v)$. Two integrations by parts imply that

$$\begin{aligned}
(Lu, v) &= \sum_{i,j=1}^{n} \int_\Omega \partial_j \left(a_{ij}\partial_i u\right) \left(\partial_k \eta^2 \partial_k u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int_\Omega \partial_k \left(a_{ij}\partial_i u\right) \left(\partial_j \eta^2 \partial_k u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int_\Omega \eta^2 a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_j \partial_k u\right) \, dx + F
\end{aligned}$$

where

$$\begin{aligned}
F = \sum_{i,j=1}^{n} \int_\Omega \Big\{ &\eta^2 \left(\partial_k a_{ij}\right) \left(\partial_i u\right) \left(\partial_j \partial_k u\right) \\
&+ 2\eta \partial_j \eta \Big[a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_k u\right) + \left(\partial_k a_{ij}\right) \left(\partial_i u\right) \left(\partial_k u\right) \Big] \Big\} \, dx.
\end{aligned}$$

The term F is linear in the second derivatives of u. We use the uniform ellipticity of L to get

$$\theta \int_{\Omega'} |D\partial_k u|^2 \, dx \leq \sum_{i,j=1}^{n} \int_{\Omega} \eta^2 a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_j \partial_k u\right) \, dx = (f, v) - F,$$

and a Cauchy inequality with ϵ to absorb the linear terms in second derivatives on the right-hand side into the quadratic terms on the left-hand side. This results in an estimate of the form

$$\|D\partial_k u\|^2_{L^2(\Omega')} \leq C \left(f^2 + \|u\|^2_{H^1(\Omega)} \right).$$

The proof of regularity is entirely analogous, with the derivatives in the test function (4.36) replaced by difference quotients (see Section 4.C). We obtain an $L^2(\Omega')$-bound for the difference quotients $D\partial_k^h u$ that is uniform in h, which implies that $u \in H^2(\Omega')$.

THEOREM 4.27. *Suppose that Ω is an open set in $\mathbb{R}^n$. Assume that $a_{ij} \in C^1(\Omega)$ and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Furthermore,*

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right) \tag{4.37}$$

where the constant C depends only on n, Ω', Ω and a_{ij}.

PROOF. Choose a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on Ω'. We use the compactly supported test function

$$v = -D_k^{-h}\left(\eta^2 D_k^h u\right) \in H_0^1(\Omega)$$

in the definition (4.34)–(4.35) for weak solutions. (As in (4.36), v is given by a positive self-adjoint operator acting on u.) This implies that

$$-\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \left(\partial_i u\right) D_k^{-h} \partial_j \left(\eta^2 D_k^h u\right) \, dx = -\int_{\Omega} f D_k^{-h} \left(\eta^2 D_k^h u\right) \, dx. \tag{4.38}$$

Performing a discrete integration by parts and using the product rule, we may write the left-hand side of (4.38) as

(4.39)
$$\begin{aligned} -\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \left(\partial_i u\right) D_k^{-h} \partial_j \left(\eta^2 D_k^h u\right) \, dx &= \sum_{i,j=1}^{n} \int_{\Omega} D_k^h \left(a_{ij} \partial_i u\right) \partial_j \left(\eta^2 D_k^h u\right) \, dx \\ &= \sum_{i,j=1}^{n} \int_{\Omega} \eta^2 a_{ij} \left(D_k^h \partial_i u\right) \left(D_k^h \partial_j u\right) \, dx + F, \end{aligned}$$

where, with $a_{ij}^h(x) = a_{ij}(x + he_k)$,

$$\begin{aligned} F = \sum_{i,j=1}^{n} \int_{\Omega} \Big\{ &\eta^2 \left(D_k^h a_{ij}\right) \left(\partial_i u\right) \left(D_k^h \partial_j u\right) \\ &+ 2\eta \partial_j \eta \Big[a_{ij}^h \left(D_k^h \partial_i u\right) \left(D_k^h u\right) + \left(D_k^h a_{ij}\right) \left(\partial_i u\right) \left(D_k^h u\right) \Big] \Big\} \, dx. \end{aligned} \tag{4.40}$$

Using the uniform ellipticity of L in (4.16), we estimate

$$\theta \int_\Omega \eta^2 \left|D_k^h Du\right|^2 \, dx \leq \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij} \left(D_k^h \partial_i u\right)\left(D_k^h \partial_j u\right) \, dx.$$

Using (4.38)–(4.39) and this inequality, we find that

$$\theta \int_\Omega \eta^2 \left|D_k^h Du\right|^2 \, dx \leq -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right) \, dx - F. \tag{4.41}$$

By the Cauchy-Schwartz inequality,

$$\left|\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right) \, dx\right| \leq \|f\|_{L^2(\Omega)} \left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)}.$$

Since $\operatorname{spt}\eta \Subset \Omega$, Proposition 4.52 implies that for sufficiently small h,

$$\begin{aligned}
\left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} &\leq \left\|\partial_k\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} \\
&\leq \left\|\eta^2 \partial_k D_k^h u\right\|_{L^2(\Omega)} + \left\|2\eta\left(\partial_k \eta\right) D_k^h u\right\|_{L^2(\Omega)} \\
&\leq \left\|\eta \partial_k D_k^h u\right\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)}.
\end{aligned}$$

A similar estimate of F in (4.40) gives

$$|F| \leq C\Big(\|Du\|_{L^2(\Omega)} \left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2\Big).$$

Using these results in (4.41), we find that

$$\begin{aligned}
\theta \left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 \leq & C\Big(\|f\|_{L^2(\Omega)} \left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|Du\|_{L^2(\Omega)} \\
& + \|Du\|_{L^2(\Omega)} \left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2\Big).
\end{aligned} \tag{4.42}$$

By Cauchy's inequality with ϵ, we have

$$\begin{aligned}
\|f\|_{L^2(\Omega)} \left\|\eta D_k^h Du\right\|_{L^2(\Omega)} &\leq \epsilon \left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon}\|f\|_{L^2(\Omega)}^2, \\
\|Du\|_{L^2(\Omega)} \left\|\eta D_k^h Du\right\|_{L^2(\Omega)} &\leq \epsilon \left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon}\|Du\|_{L^2(\Omega)}^2.
\end{aligned}$$

Hence, choosing ϵ so that $4C\epsilon = \theta$, and using the result in (4.42) we get that

$$\frac{\theta}{4}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 \leq C\Big(\|f\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\Big).$$

Thus, since $\eta = 1$ on Ω',

$$\left\|D_k^h Du\right\|_{L^2(\Omega')}^2 \leq C\Big(\|f\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\Big) \tag{4.43}$$

where the constant C depends on Ω, Ω', a_{ij}, but is independent of h, u, f.

We can further estimate $\|Du\|$ in terms of $\|u\|$ by taking $v = u$ in (4.34)–(4.35) and using the uniform ellipticity of L to get

$$\begin{aligned}\theta \int_\Omega |Du|^2\,dx &\le \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j u \\ &\le \int_\Omega f u\,dx \\ &\le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Using this result in (4.43), we get that

$$\left\|D_k^h Du\right\|^2_{L^2(\Omega')} \le C\Big(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big).$$

Theorem 4.53 theorem now implies that the weak second derivatives of u exist and belong to $L^2(\Omega)$. Furthermore, the H^2-norm of u satisfies (4.37). $\square$

If $u \in H^2_{\mathrm{loc}}(\Omega)$ and $f \in L^2(\Omega)$, then the equation $Lu = f$ relating the weak derivatives of u and f holds pointwise a.e.; such solutions are often called strong solutions, to distinguish them from weak solutions which may not possess weak second order derivatives and classical solutions which possess continuous second order derivatives.

The repeated application of these estimates leads to higher interior regularity.

Theorem 4.28. *Suppose that $a_{ij} \in C^{k+1}(\Omega)$ and $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^{k+2}(\Omega')$ for every $\Omega' \Subset \Omega$. Furthermore,*

$$\|u\|_{H^{k+2}(\Omega')} \le C\left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, k, Ω', Ω and a_{ij}.

See [**5**] for a detailed proof. Note that if the above conditions hold with $k > n/2$, then $f \in C(\Omega)$ and $u \in C^2(\Omega)$, so u is a classical solution of the PDE $Lu = f$. Furthermore, if f and a_{ij} are smooth then so is the solution.

Corollary 4.29. *If $a_{ij}, f \in C^\infty(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in C^\infty(\Omega)$*

Proof. If $\Omega' \Subset \Omega$, then $f \in H^k(\Omega')$ for every $k \in \mathbb{N}$, so by Theorem (4.28) $u \in H^{k+2}_{\mathrm{loc}}(\Omega')$ for every $k \in \mathbb{N}$, and by the Sobolev imbedding theorem $u \in C^\infty(\Omega')$. Since this holds for every open set $\Omega' \Subset \Omega$, we have $u \in C^\infty(\Omega)$. $\square$

4.12. Boundary regularity

To study the regularity of solutions near the boundary, we localize the problem to a neighborhood of a boundary point by use of a partition of unity: We decompose the solution into a sum of functions that are compactly supported in the sets of a suitable open cover of the domain and estimate each function in the sum separately.

Assuming, as in Section 1.10, that the boundary is at least C^1, we may 'flatten' the boundary in a neighborhood U by a diffeomorphism $\varphi : U \to V$ that maps $U \cap \Omega$ to an upper half space $V = B_1(0) \cap \{y_n > 0\}$. If $\varphi^{-1} = \psi$ and $x = \psi(y)$, then by a change of variables (*c.f.* Theorem 1.38 and Proposition 3.20) the weak formulation (4.32)–(4.33) on U becomes

$$\sum_{i,j=1}^{n} \int_V \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial y_i} \frac{\partial \tilde{v}}{\partial y_j}\, dy = \int_V \tilde{f} \tilde{v}\, dy \qquad \text{for all functions } \tilde{v} \in H_0^1(V),$$

where $\tilde{u} \in H^1(V)$. Here, $\tilde{u} = u \circ \psi$, $\tilde{v} = v \circ \psi$, and

$$\tilde{a}_{ij} = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \left(\frac{\partial \varphi_i}{\partial x_p} \circ \psi\right) \left(\frac{\partial \varphi_j}{\partial x_q} \circ \psi\right), \qquad \tilde{f} = |\det D\psi|\, f \circ \psi.$$

The matrix $\tilde{a}_{ij}$ satisfies the uniform ellipticity condition if a_{pq} does. To see this, we define $\zeta = (D\varphi^t)\, \xi$, or

$$\zeta_p = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_p} \xi_i.$$

Then, since $D\varphi$ and $D\psi = D\varphi^{-1}$ are invertible and bounded away from zero, we have for some constant $C > 0$ that

$$\sum_{i,j}^{n} \tilde{a}_{ij} \xi_i \xi_j = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \zeta_p \zeta_q \geq |\det D\psi|\ \theta |\zeta|^2 \geq C\theta |\xi|^2.$$

Thus, we obtain a problem of the same form as before after the change of variables. Note that we must require that the boundary is C^2 to ensure that $\tilde{a}_{ij}$ is C^1.

It is important to recognize that in changing variables for weak solutions, we need to verify the change of variables for the weak formulation directly and not for the original PDE. A transformation that is valid for smooth solutions of a PDE is not always valid for weak solutions, which may lack sufficient smoothness to justify the transformation.

We now state a boundary regularity theorem. Unlike the interior regularity theorem, we impose a boundary condition $u \in H_0^1(\Omega)$ on the solution, and we require that the boundary of the domain is smooth. A solution of an elliptic PDE with smooth coefficients and smooth right-hand side is smooth in the interior of its domain of definition, whatever its behavior near the boundary; but we cannot expect to obtain smoothness up to the boundary without imposing a smooth boundary condition on the solution and requiring that the boundary is smooth.

THEOREM 4.30. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^2-boundary. Assume that $a_{ij} \in C^1(\overline{\Omega})$ and $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^2(\Omega)$, and*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

where the constant C depends only on n, Ω and a_{ij}.

PROOF. By use of a partition of unity and a flattening of the boundary, it is sufficient to prove the result for an upper half space $\Omega = \{(x_1, \ldots, x_n) : x_n > 0\}$ space and functions $u, f : \Omega \to \mathbb{R}$ that are compactly supported in $B_1(0) \cap \overline{\Omega}$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_1(0)$. We will estimate the tangential and normal difference quotients of Du separately.

First consider a test function that depends on tangential differences,

$$v = -D_k^{-h} \eta^2 D_k^h u \qquad \text{for } k = 1, 2, \ldots, n-1.$$

Since the trace of u is zero on $\partial\Omega$, the trace of v on $\partial\Omega$ is zero and, by Theorem 3.42, $v \in H_0^1(\Omega)$. Thus we may use v in the definition of weak solution to get (4.38). Exactly the same argument as the one in the proof of Theorem 4.27 gives (4.43). It follows from Theorem 4.53 that the weak derivatives $\partial_k \partial_i u$ exist and satisfy

$$\|\partial_k Du\|_{L^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right) \qquad \text{for } k = 1, 2, \dots, n-1. \tag{4.44}$$

The only derivative that remains is the second-order normal derivative $\partial_n^2 u$, which we can estimate from the equation. Using (4.32)–(4.33), we have for $\phi \in C_c^\infty(\Omega)$ that

$$\int_\Omega a_{nn} \left(\partial_n u\right) \left(\partial_n \phi\right)\, dx = -{\sum}' \int_\Omega a_{ij} \left(\partial_i u\right) \left(\partial_j \phi\right)\, dx + \int_\Omega f\phi\, dx$$

where ${\sum}'$ denotes the sum over $1 \leq i, j \leq n$ with the term $i = j = n$ omitted. Since $a_{ij} \in C^1(\Omega)$ and $\partial_i u$ is weakly differentiable with respect to x_j unless $i = j = n$ we get, using Proposition 3.20, that

$$\int_\Omega a_{nn} \left(\partial_n u\right) \left(\partial_n \phi\right)\, dx = {\sum}' \int_\Omega \left\{ \partial_j \left[a_{ij} \left(\partial_i u\right)\right] + f \right\} \phi\, dx \qquad \text{for every } \phi \in C_c^\infty(\Omega).$$

It follows that $a_{nn} \left(\partial_n u\right)$ is weakly differentiable with respect to x_n, and

$$\partial_n \left[a_{nn} \left(\partial_n u\right)\right] = -\left\{ {\sum}' \partial_j \left[a_{ij} \left(\partial_i u\right)\right] + f \right\} \in L^2(\Omega).$$

From the uniform ellipticity condition (4.16) with $\xi = e_n$, we have $a_{nn} \geq \theta$. Hence, by Proposition 3.20,

$$\partial_n u = \frac{1}{a_{nn}}\, a_{nn} \partial_n u$$

is weakly differentiable with respect to x_n with derivative

$$\partial_{nn}^2 u = \frac{1}{a_{nn}} \partial_n \left[a_{nn} \partial_n u\right] + \partial_n \left(\frac{1}{a_{nn}} \right) a_{nn} \partial_n u \in L^2(\Omega).$$

Furthermore, using (4.44) we get an estimate of the same form for $\|\partial_{nn}^2 u\|_{L^2(\Omega)}^2$, so that

$$\left\| D^2 u \right\|_{L^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right)$$

$\square$

The repeated application of these estimates leads to higher-order regularity.

Theorem 4.31. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^{k+2}-boundary. Assume that $a_{ij} \in C^{k+1}(\overline{\Omega})$ and $f \in H^k(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in H^{k+2}(\Omega)$ and*

$$\|u\|_{H^{k+2}(\Omega)} \leq C \left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

where the constant C depends only on n, k, Ω, and a_{ij}.

Sobolev imbedding then yields the following result.

Corollary 4.32. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^∞ boundary. If $a_{ij}, f \in C^\infty(\overline{\Omega})$ and $u \in H_0^1(\Omega)$ is a weak solution of (4.32)–(4.33), then $u \in C^\infty(\overline{\Omega})$*

4.13. Some further perspectives

The above results give an existence and L^2-regularity theory for second-order, uniformly elliptic PDEs in divergence form. This theory is based on the simple *a priori* energy estimate for $\|Du\|_{L^2}$ that we obtain by multiplying the equation $Lu = f$ by u, or some derivative of u, and integrating the result by parts.

This theory is a fundamental one, but there is a bewildering variety of approaches to the existence and regularity of solutions of elliptic PDEs. In an attempt to put the above analysis in a broader context, we briefly list some of these approaches and other important results, without any claim to completeness. Many of these topics are discussed further in the references [**5, 10, 12**].

L^p-theory: If $1 < p < \infty$, there is a similar regularity result that solutions of $Lu = f$ satisfy $u \in W^{2,p}$ if $f \in L^p$. The derivation is not as simple when $p \neq 2$, however, and requires the use of more sophisticated tools from real analysis (such as Calderón-Zygmund operators in harmonic analysis).

Schauder theory: The Schauder theory provides Hölder-estimates similar to those derived in Section 2.7.2 for Laplace's equation, and a corresponding existence theory of solutions $u \in C^{2,\alpha}$ of $Lu = f$ if $f \in C^{0,\alpha}$ and L has Hölder continuous coefficients. General linear elliptic PDEs are treated by regarding them as perturbations of constant coefficient PDEs, an approach that works because there is no 'loss of derivatives' in the estimates of the solution. The Hölder estimates were originally obtained by the use of potential theory, but other ways to obtain them are now known; for example, by the use of Campanato spaces, which provide Hölder norms in terms of suitable integrals that are easier to estimate directly.

Perron's method: Perron (1923) showed that solutions of the Dirichlet problem for Laplace's equation can be obtained as the infimum of superharmonic functions or the supremum of subharmonic functions, together with the use of barrier functions to prove that, under suitable assumptions on the boundary, the solution attains the prescribed boundary values. This method is based on maximum principle estimates.

Boundary integral methods: By the use of Green's functions, one can often reduce a linear elliptic BVP to an integral equation on the boundary, and then use the theory of integral equations to study the existence and regularity of solutions. These methods also provide efficient numerical schemes because of the lower dimensionality of the boundary.

Pseudo-differential operators: The Fourier transform provides an effective method for solving linear PDEs with constant coefficients. The theory of pseudo-differential and Fourier-integral operators is a powerful extension of this method that applies to general linear PDEs with variable coefficients, and elliptic PDEs in particular. It is, however, less well-suited to the analysis of nonlinear PDEs.

Variational methods: Many elliptic PDEs — especially those in divergence form — arise as Euler-Lagrange equations for variational principles. Existence of weak solutions can often be shown by use of the direct method of the calculus of variations, after which one studies the regularity of a minimizer (or, in some cases, a critical point).

Di Giorgi-Nash-Moser: The work of Di Giorgi (1957), Nash (1958), and Moser (1960) showed that weak solutions of a second order elliptic PDE in divergence form with bounded (L^∞) coefficients are Hölder continuous ($C^{0,\alpha}$). This was the key step in developing a regularity theory for minimizers of nonlinear variational principles with elliptic Euler-Lagrange equations. Moser also obtained a Harnack inequality for weak solutions.

Fully nonlinear equations: Krylov and Safonov (1979) obtained a Harnack inequality for second order elliptic equations in nondivergence form. This allowed the development of a regularity theory for fully nonlinear elliptic equations (*e.g.* second-order equations for u that depend nonlinearly on D^2u). Crandall and Lions (1983) introduced the notion of viscosity solutions which — despite the name — uses the maximum principle and is based on a comparison with appropriate sub and super solutions This theory applies to fully nonlinear elliptic PDEs, although it is mainly restricted to scalar equations.

Degree theory: Topological methods based on the Leray-Schauder degree of a mapping on a Banach space can be used to prove existence of solutions of various nonlinear elliptic problems (see *e.g.* L. Nirenberg, *Topics in Nonlinear Functional Analysis*). These methods can provide global existence results for large solutions, but often do not give much detailed analytical information about the solutions.

Heat flow methods: Parabolic PDEs, such as $u_t + Lu = f$, are closely connected with the associated elliptic PDEs for stationary solutions, such as $Lu = f$. One may use this connection to obtain solutions of an elliptic PDE as the limit as $t \to \infty$ of solutions of the associated parabolic PDE. For example, Hamilton (1981) introduced the Ricci flow on a manifold, in which the metric approaches a Ricci-flat metric as $t \to \infty$, as a means to understand the topological classification of smooth manifolds, and Perelman (2003) used this approach to prove the Poincaré conjecture (that every simply connected, three-dimensional, compact manifold without boundary is homeomorphic to a three-dimensional sphere) and, more generally, the geometrization conjecture of Thurston.

Appendix

4.A. Heat flow

As a simple application that leads to second order PDEs, we consider the problem of finding the temperature distribution inside a body. Similar equations describe the diffusion of a solute. Steady temperature distributions satisfy an elliptic PDE, such as Laplace's equation, while unsteady distributions satisfy a parabolic PDE, such as the heat equation.

4.A.1. Steady heat flow. Suppose that the body occupies an open set Ω in $\mathbb{R}^n$. Let $u : \Omega \to \mathbb{R}$ denote the temperature, $g : \Omega \to \mathbb{R}$ the rate per unit volume at which heat sources create energy inside the body, and $\vec{q} : \Omega \to \mathbb{R}^n$ the heat flux. That is, the rate per unit area at which heat energy diffuses across a surface with normal $\vec{\nu}$ is equal to $\vec{q} \cdot \vec{\nu}$.

If the temperature distribution is steady, then conservation of energy implies that for any smooth open set $\Omega' \Subset \Omega$ the heat flux out of Ω' is equal to the rate at which heat energy is generated inside Ω'; that is,

$$\int_{\partial\Omega'} \vec{q} \cdot \vec{\nu}\, dS = \int_{\Omega'} g\, dV.$$

Here, we use dS and dV to denote integration with respect to surface area and volume, respectively.

We assume that $\vec{q}$ and g are smooth. Then, by the divergence theorem,

$$\int_{\Omega'} \operatorname{div} \vec{q}\, dV = \int_{\Omega'} g\, dV.$$

Since this equality holds for all subdomains Ω' of Ω, it follows that

$$\operatorname{div} \vec{q} = g \qquad \text{in } \Omega. \tag{4.45}$$

Equation (4.45) expresses the fundamental physical principle of conservation of energy, but this principle alone is not enough to determine the temperature distribution inside the body. We must supplement it with a constitutive relation that describes how the heat flux is related to the temperature distribution.

Fourier's law states that the heat flux at some point of the body depends linearly on the temperature gradient at the same point and is in a direction of decreasing temperature. This law is an excellent and well-confirmed approximation in a wide variety of circumstances. Thus,

$$\vec{q} = -A\nabla u \tag{4.46}$$

for a suitable conductivity tensor $A : \Omega \to L(\mathbb{R}^n, \mathbb{R}^n)$, which is required to be symmetric and positive definite. Explicitly, if $\vec{x} \in \Omega$, then $A(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map that takes the negative temperature gradient at $\vec{x}$ to the heat flux at $\vec{x}$. In a uniform, isotropic medium $A = \kappa I$ where the constant $\kappa > 0$ is the thermal conductivity. In an anisotropic medium, such as a crystal or a composite medium, A is not proportional to the identity I and the heat flux need not be in the same direction as the temperature gradient.

Using (4.46) in (4.45), we find that the temperature u satisfies

$$-\operatorname{div}(A\nabla u) = g.$$

If we denote the matrix of A with respect to the standard basis in $\mathbb{R}^n$ by (a_{ij}), then the component form of this equation is

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) = g.$$

This equation is in divergence or conservation form. For smooth functions $a_{ij} : \Omega \to \mathbb{R}$, we can write it in nondivergence form as

$$-\sum_{i,j=1}^{n} a_{ij}\partial_{ij}u - \sum_{j=1}^{n} b_j\partial_j u = g, \qquad b_j = \sum_{i}^{n} \partial_i a_{ij}.$$

These forms need not be equivalent if the coefficients a_{ij} are not smooth. For example, in a composite medium made up of different materials, a_{ij} may be discontinuous across boundaries that separate the materials. Such problems can be rewritten as smooth PDEs within domains occupied by a given material, together with appropriate jump conditions across the boundaries. The weak formulation incorporates both the PDEs and the jump conditions.

Next, suppose that the body is occupied by a fluid which, in addition to conducting heat, is in motion with velocity $\vec{v} : \Omega \to \mathbb{R}^n$. Let $e : \Omega \to \mathbb{R}$ denote the internal thermal energy per unit volume of the body, which we assume is a function of the location $\vec{x} \in \Omega$ of a point in the body. Then, in addition to the diffusive flux $\vec{q}$, there is a convective thermal energy flux equal to $e\vec{v}$, and conservation of energy gives

$$\int_{\partial\Omega'} (\vec{q} + e\vec{v}) \cdot \vec{\nu}\, dS = \int_{\Omega'} g\, dV.$$

Using the divergence theorem as before, we find that

$$\operatorname{div}(\vec{q} + e\vec{v}) = g,$$

If we assume that $e = c_p u$ is proportional to the temperature, where c_p is the heat capacity per unit volume of the material in the body, and Fourier's law, we get the PDE

$$-\operatorname{div}(A\nabla u) + \operatorname{div}(\vec{b}u) = g.$$

where $\vec{b} = c_p\vec{v}$.

Suppose that $g = f - cu$ where $f : \Omega \to \mathbb{R}$ is a given energy source and cu represents a linear growth or decay term with coefficient $c : \Omega \to \mathbb{R}$. For example, lateral heat loss at a rate proportional the temperature would give decay $(c > 0)$, while the effects of an exothermic temperature-dependent chemical reaction might be approximated by a linear growth term $(c < 0)$. We then get the linear PDE

$$-\operatorname{div}(A\nabla u) + \operatorname{div}(\vec{b}u) + cu = f,$$

or in component form with $\vec{b} = (b_1, \dots, b_n)$

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu = f.$$

This PDE describes a thermal equilibrium due to the combined effects of diffusion with diffusion matrix a_{ij}, advection with normalized velocity b_i, growth or decay with coefficient c, and external sources with density f.

In the simplest case where, after nondimensionalization, $A = I$, $\vec{b} = 0$, $c = 0$, and $f = 0$, we get Laplace's equation $\Delta u = 0$.

4.A.2. Unsteady heat flow. Consider a time-dependent heat flow in a region Ω with temperature $u(\vec{x},t)$, energy density per unit volume $e(\vec{x},t)$, heat flux $\vec{q}(\vec{x},t)$, advection velocity $\vec{v}(\vec{x},t)$, and heat source density $g(\vec{x},t)$. Conservation of energy implies that for any subregion $\Omega' \Subset \Omega$

$$\frac{d}{dt}\int_{\Omega'} e\,dV = -\int_{\partial\Omega'} (\vec{q}+e\vec{v})\cdot\vec{\nu}\,dS + \int_{\Omega'} g\,dV.$$

Since

$$\frac{d}{dt}\int_{\Omega'} e\,dV = \int_{\Omega'} e_t\,dV,$$

the use of the divergence theorem and the same constitutive assumptions as in the steady case lead to the parabolic PDE

$$(c_p u)_t - \sum_{i,j=1}^{n} \partial_i\left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n}\partial_i\left(b_i u\right) + cu = f.$$

In the simplest case where, after nondimensionalization, $c_p = 1$, $A = I$, $\vec{b} = 0$, $c = 0$, and $f = 0$, we get the heat equation $u_t = \Delta u$.

4.B. Operators on Hilbert spaces

Suppose that $\mathcal{H}$ is a Hilbert space with inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$. We denote the space of bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. This is a Banach space with respect to the operator norm, defined by

$$\|T\| = \sup_{\substack{x\in\mathcal{H}\\ x\neq 0}} \frac{\|Tx\|}{\|x\|}.$$

The adjoint $T^* \in \mathcal{B}(\mathcal{H})$ of $T \in \mathcal{B}(\mathcal{H})$ is the linear operator such that

$$(Tx, y) = (x, T^*y) \qquad \text{for all } x, y \in \mathcal{H}.$$

An operator T is self-adjoint if $T = T^*$. The kernel and range of $T \in \mathcal{B}(\mathcal{H})$ are the subspaces

$$\ker T = \{x \in \mathcal{H} : Tx = 0\}, \quad \operatorname{ran} T = \{y \in \mathcal{H} : y = Tx \text{ for some } x \in \mathcal{H}\}.$$

We denote by $\ell^2(\mathbb{N})$, or ℓ^2 for short, the Hilbert space of square summable real sequences

$$\ell^2(\mathbb{N}) = \left\{(x_1, x_2, x_3, \dots, x_n, \dots) : x_n \in \mathbb{R} \text{ and } \textstyle\sum_{n\in\mathbb{N}} x_n^2 < \infty\right\}$$

with the standard inner product. Any infinite-dimensional, separable Hilbert space is isomorphic to ℓ^2.

4.B.1. Compact operators.

Definition 4.33. A linear operator $T \in \mathcal{B}(\mathcal{H})$ is compact if it maps bounded sets to precompact sets.

That is, T is compact if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in $\mathcal{H}$.

Example 4.34. A bounded linear map with finite-dimensional range is compact. In particular, every linear operator on a finite-dimensional Hilbert space is compact.

EXAMPLE 4.35. The identity map $I \in \mathcal{B}(\mathcal{H})$ given by $I : x \mapsto x$ is compact if and only if $\mathcal{H}$ is finite-dimensional.

EXAMPLE 4.36. The map $K \in \mathcal{B}\left(\ell^2\right)$ given by

$$K : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots, \frac{1}{n}x_n, \dots\right)$$

is compact (and self-adjoint).

We have the following spectral theorem for compact self-adjoint operators.

THEOREM 4.37. *Let $T : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator. Then T has a finite or countably infinite number of distinct nonzero, real eigenvalues. If there are infinitely many eigenvalues $\{\lambda_n \in \mathbb{R} : n \in \mathbb{N}\}$ then $\lambda_n \to 0$ as $n \to \infty$. The eigenspace associated with each nonzero eigenvalue is finite-dimensional, and eigenvectors associated with distinct eigenvalues are orthogonal. Furthermore, $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T, including those (if any) with eigenvalue zero.*

4.B.2. Fredholm operators. We summarize the definition and properties of Fredholm operators and give some examples. For proofs, see

DEFINITION 4.38. A linear operator $T \in \mathcal{B}(\mathcal{H})$ is Fredholm if: (a) $\ker T$ has finite dimension; (b) $\operatorname{ran} T$ is closed and has finite codimension.

Condition (b) and the projection theorem for Hilbert spaces imply that $\mathcal{H} = \operatorname{ran} T \oplus (\operatorname{ran} T)^\perp$ where the dimension of $\operatorname{ran} T^\perp$ is finite, and

$$\operatorname{codim} \operatorname{ran} T = \dim(\operatorname{ran} T)^\perp.$$

DEFINITION 4.39. If $T \in \mathcal{B}(\mathcal{H})$ is Fredholm, then the index of T is the integer

$$\operatorname{ind} T = \dim \ker T - \operatorname{codim} \operatorname{ran} T.$$

EXAMPLE 4.40. Every linear operator $T : \mathcal{H} \to \mathcal{H}$ on a finite-dimensional Hilbert space $\mathcal{H}$ is Fredholm and has index zero. The range is closed since every finite-dimensional linear space is closed, and the dimension formula

$$\dim \ker T + \dim \operatorname{ran} T = \dim \mathcal{H}$$

implies that the index is zero.

EXAMPLE 4.41. The identity map I on a Hilbert space of any dimension is Fredholm, with $\dim \ker P = \operatorname{codim} \operatorname{ran} P = 0$ and $\operatorname{ind} I = 0$.

EXAMPLE 4.42. The self-adjoint projection P on ℓ^2 given by

$$P : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_2, x_3, \dots, x_n, \dots)$$

is Fredholm, with $\dim \ker P = \operatorname{codim} \operatorname{ran} P = 1$ and $\operatorname{ind} P = 0$. The complementary projection

$$Q : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_1, 0, 0, \dots, 0, \dots)$$

is not Fredholm, although the range of Q is closed, since $\dim \ker Q$ and $\operatorname{codim} \operatorname{ran} Q$ are infinite.

EXAMPLE 4.43. The left and right shift maps on ℓ^2, given by

$$S : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_2, x_3, x_4, \dots, x_{n+1}, \dots),$$
$$T : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_1, x_2, \dots, x_{n-1}, \dots),$$

are Fredholm. Note that $S^* = T$. We have $\dim\ker S = 1$, $\operatorname{codim}\operatorname{ran} S = 0$, and $\dim\ker T = 0$, $\operatorname{codim}\operatorname{ran} T = 1$, so

$$\operatorname{ind} S = 1, \qquad \operatorname{ind} T = -1.$$

If $n \in \mathbb{N}$, then $\operatorname{ind} S^n = n$ and $\operatorname{ind} T^n = -n$, so the index of a Fredholm operator on an infinite-dimensional space can take all integer values. Unlike the finite-dimensional case, where a linear operator $A : \mathcal{H} \to \mathcal{H}$ is one-to-one if and only if it is onto, S fails to be one-to-one although it is onto, and T fails to be onto although it is one-to-one.

The above example also illustrates the following theorem.

THEOREM 4.44. *If $T \in \mathcal{B}(\mathcal{H})$ is Fredholm, then T^* is Fredholm with*

$$\dim\ker T^* = \operatorname{codim}\operatorname{ran} T, \quad \operatorname{codim}\operatorname{ran} T^* = \dim\ker T, \quad \operatorname{ind} T^* = -\operatorname{ind} T.$$

EXAMPLE 4.45. The compact map K in Example 4.36 is not Fredholm since the range of K,

$$\operatorname{ran} K = \left\{ (y_1, y_2, y_3, \dots, y_n, \dots) \in \ell^2 : \sum_{n\in\mathbb{N}} n^2 y_n^2 < \infty \right\},$$

is not closed. The range is dense in ℓ^2 but, for example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2 \setminus \operatorname{ran} K.$$

We denote the set of Fredholm operators by $\mathcal{F}$. Then, according to the next theorem, $\mathcal{F}$ is an open set in $\mathcal{B}(\mathcal{H})$, and

$$\mathcal{F} = \bigcup_{n\in\mathbb{Z}} \mathcal{F}_n$$

is the union of connected components $\mathcal{F}_n$ consisting of the Fredholm operators with index n. Moreover, if $T \in \mathcal{F}_n$, then $T + K \in \mathcal{F}_n$ for any compact operator K.

THEOREM 4.46. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is Fredholm and $K \in \mathcal{B}(\mathcal{H})$ is compact.*

(1) *There exists $\epsilon > 0$ such that $T + H$ is Fredholm for any $H \in \mathcal{B}(\mathcal{H})$ with $\|H\| < \epsilon$. Moreover,* $\operatorname{ind}(T + H) = \operatorname{ind} T$.
(2) *$T + K$ is Fredholm and* $\operatorname{ind}(T + K) = \operatorname{ind} T$.

Solvability conditions for Fredholm operators are a consequences of following theorem.

THEOREM 4.47. *If $T \in \mathcal{B}(\mathcal{H})$, then $\mathcal{H} = \overline{\operatorname{ran} T} \oplus \ker T^*$ and $\overline{\operatorname{ran} T} = (\ker T)^\perp$.*

Thus, if $T \in \mathcal{B}(\mathcal{H})$ has closed range, then $Tx = y$ has a solution $x \in \mathcal{H}$ if and only if $y \perp z$ for every $z \in \mathcal{H}$ such that $T^* z = 0$. For a Fredholm operator, this is finitely many linearly independent solvability conditions.

EXAMPLE 4.48. If S, T are the shift maps defined in Example 4.43, then $\ker S^* = \ker T = 0$ and the equation $Sx = y$ is solvable for every $y \in \ell^2$. Solutions are not, however, unique since $\ker S \neq 0$. The equation $Tx = y$ is solvable only if $y \perp \ker S$. If it exists, the solution is unique.

EXAMPLE 4.49. The compact map K in Example 4.36 is self adjoint, $K = K^*$, and $\ker K = 0$. Thus, every element $y \in \ell^2$ is orthogonal to $\ker K^*$, but this condition is not sufficient to imply the solvability of $Kx = y$ because the range of K os not closed. For example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2 \setminus \operatorname{ran} K.$$

For Fredholm operators with index zero, we get the following Fredholm alternative, which states that the corresponding linear equation has solvability properties which are similar to those of a finite-dimensional linear system.

THEOREM 4.50. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator and* $\operatorname{ind} T = 0$. *Then one of the following two alternatives holds:*

(1) $\ker T^* = 0$*;* $\ker T = 0$*;* $\operatorname{ran} T = \mathcal{H}$*,* $\operatorname{ran} T^* = \mathcal{H}$*;*
(2) $\ker T^* \neq 0$*;* $\ker T$*,* $\ker T^*$ *are finite-dimensional spaces with the same dimension;* $\operatorname{ran} T = (\ker T^*)^\perp$*,* $\operatorname{ran} T^* = (\ker T)^\perp$*.*

4.C. Difference quotients and weak derivatives

Difference quotients provide a useful method for proving the weak differentiability of functions. The main result, in Theorem 4.53 below, is that the uniform boundedness of the difference quotients of a function is sufficient to imply that the function is weakly differentiable.

DEFINITION 4.51. If $u : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R} \setminus \{0\}$, the ith difference quotient of u of size h is the function $D_i^h u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

where e_i is the unit vector in the ith direction. The vector of difference quotient is

$$D^h u = \left(D_1^h u, D_2^h u, \dots, D_n^h u\right).$$

The next proposition gives some elementary properties of difference quotients that are analogous to those of derivatives.

PROPOSITION 4.52. *The difference quotient has the following properties.*

(1) *Commutativity with weak derivatives: if* $u, \partial_i u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*, then*

$$\partial_i D_j^h u = D_j^h \partial_i u.$$

(2) *Integration by parts: if* $u \in L^p(\mathbb{R}^n)$ *and* $v \in L^{p'}(\mathbb{R}^n)$*, where* $1 \leq p \leq \infty$*, then*

$$\int (D_i^h u) v \, dx = -\int u (D_i^h v) \, dx.$$

(3) *Product rule:*

$$D_i^h(uv) = u_i^h \left(D_i^h v\right) + \left(D_i^h u\right) v = u \left(D_i^h v\right) + \left(D_i^h u\right) v_i^h.$$

where $u_i^h(x) = u(x + he_i)$*.*

PROOF. Property (1) follows immediately from the linearity of the weak derivative. For (2), note that

$$\begin{aligned}
\int (D_i^h u) v \, dx &= \frac{1}{h} \int [u(x+he_i) - u(x)] \, v(x) \, dx \\
&= \frac{1}{h} \int u(x') v(x' - he_i) \, dx' - \frac{1}{h} \int u(x) v(x) \, dx \\
&= \frac{1}{h} \int u(x) \left[v(x - he_i) - v(x) \right] \, dx \\
&= - \int u \left(D_i^{-h} v \right) \, dx.
\end{aligned}$$

For (3), we have

$$\begin{aligned}
u_i^h \left(D_i^h v \right) + \left(D_i^h u \right) v &= u(x+he_i) \left[\frac{v(x+he_i) - v(x)}{h} \right] + \left[\frac{u(x+he_i) - u(x)}{h} \right] v(x) \\
&= \frac{u(x+he_i) v(x+he_i) - u(x) v(x)}{h} \\
&= D_i^h (uv),
\end{aligned}$$

and the same calculation with u and v exchanged. $\square$

THEOREM 4.53. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\Omega' \Subset \Omega$. Let*

$$d = \operatorname{dist}(\Omega', \partial\Omega) > 0.$$

(1) *If $Du \in L^p(\Omega)$ where $1 \leq p < \infty$, and $0 < |h| < d$, then*

$$\left\| D^h u \right\|_{L^p(\Omega')} \leq \| Du \|_{L^p(\Omega)} .$$

(2) *If $u \in L^p(\Omega)$ where $1 < p < \infty$, and there exists a constant C such that*

$$\left\| D^h u \right\|_{L^p(\Omega')} \leq C$$

for all $0 < |h| < d/2$, then $u \in W^{1,p}(\Omega')$ and

$$\| Du \|_{L^p(\Omega')} \leq C.$$

PROOF. To prove (1), we may assume by an approximation argument that u is smooth. Then

$$u(x+he_i) - u(x) = h \int_0^1 \partial_i u(x + te_i) \, dt,$$

and, by Jensen's inequality,

$$|u(x+he_i) - u(x)|^p \leq |h|^p \int_0^1 |\partial_i u(x+te_i)|^p \, dt.$$

Integrating this inequality with respect to x, and using Fubini's theorem, together with the fact that $x + te_i \in \Omega$ if $x \in \Omega'$ and $|t| \leq h < d$, we get

$$\int_{\Omega'} |u(x+he_i) - u(x)|^p \, dx \leq |h|^p \int_{\Omega} |\partial_i u(x+te_i)|^p \, dx.$$

Thus, $\|D_i^h u\|_{L^p(\Omega')} \leq \|D_i^h u\|_{L^p(\Omega)}$, and (1) follows.

To prove (2), note that since

$$\left\{ D_i^h u : 0 < |h| < d \right\}$$

is bounded in $L^p(\Omega')$, the Banach-Alaoglu theorem implies that there is a sequence $\{h_k\}$ such that $h_k \to 0$ as $k \to \infty$ and a function $v_i \in L^p(\Omega')$ such that

$$D_i^{h_k} u \rightharpoonup v_i \qquad \text{as } k \to \infty \text{ in } L^p(\Omega').$$

Suppose that $\phi \in C_c^\infty(\Omega')$. Then, for sufficiently small h_k,

$$\int_{\Omega'} u D_i^{-h_k} \phi \, dx = \int_{\Omega'} \left(D_i^{h_k} u \right) \phi \, dx.$$

Taking the limit as $k \to \infty$, when $D_i^{-h_k}\phi$ converges uniformly to $\partial_i \phi$, we get

$$\int_{\Omega'} u \partial_i \phi \, dx = \int_{\Omega'} v_i \phi \, dx.$$

Hence u is weakly differentiable and $\partial_i u = v_i \in L^p(\Omega')$, which proves (2). $\square$

CHAPTER 5

The Heat Equation

The heat, or diffusion, equation is

$$u_t = \Delta u. \tag{5.1}$$

Section 4.A derives (5.1) as a model of heat flow.

Steady solutions of the heat equation satisfy Laplace's equation. Using (2.4), we have for smooth functions that

$$\begin{aligned}
\Delta u(x) &= \lim_{r\to 0^+} \fint_{B_r(x)} \Delta u \, dx \\
&= \lim_{r\to 0^+} \frac{n}{r}\frac{\partial}{\partial r}\left[\fint_{\partial B_r(x)} u \, dS\right] \\
&= \lim_{r\to 0^+} \frac{2n}{r^2}\left[\fint_{\partial B_r(x)} u \, dS - u(x)\right].
\end{aligned}$$

Thus, if u is a solution of the heat equation, then the rate of change of $u(x,t)$ with respect to t at a point x is proportional to the difference between the value of u at x and the average of u over nearby spheres centered at x. The solution decreases in time if its value at a point is greater than the nearby averages and increases if its value is less than the nearby averages. The heat equation therefore describes the evolution of a function towards its mean. As $t \to \infty$ solutions of the heat equation typically approach functions with the mean value property, which are solutions of Laplace's equation.

The properties of the heat equation and more general parabolic PDEs parallel those of Laplace's equation and elliptic PDEs. For example, there are parabolic versions of maximum principles, Harnack inequalities, Schauder theory, and Sobolev solutions.

5.1. The initial value problem

Consider the initial value problem for $u(x,t)$ where $x \in \mathbb{R}^n$

$$\begin{aligned}
u_t &= \Delta u \qquad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\
u(x,0) &= f(x) \qquad \text{for } x \in \mathbb{R}^n.
\end{aligned} \tag{5.2}$$

We will solve (5.2) explicitly by use of the Fourier transform, following the presentation in [**15**]. Before doing this, we describe the sense in which we define a solution.

5.1.1. Schwartz solutions. Assume first that the initial data $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth, rapidly decreasing Schwartz function $f \in \mathcal{S}$ (see Section 5.A). The solution we construct is also a Schwartz function of x at later times $t > 0$, and we will regard it as a function of time with values in $\mathcal{S}$. This is analogous to the geometrical interpretation of a first-order system of ODEs, in which the finite-dimensional phase space of the ODE is replaced by the infinite-dimensional function space $\mathcal{S}$; we then think of a solution of the heat equation as a parametrized curve in the vector space $\mathcal{S}$. A similar viewpoint is useful for many evolutionary PDEs, where the Schwartz space may be replaced other function spaces (for example, Sobolev spaces).

By a convenient abuse of notation, we use the same symbol u to denote the scalar-valued function $u(x,t)$, where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, and the associated vector-valued function $u(t)$, where $u : [0,\infty) \to \mathcal{S}$. We write the vector-valued function corresponding to the associated scalar-valued function as $u(t) = u(\cdot, t)$.

Definition 5.1. Suppose that (a,b) is an open interval in $\mathbb{R}$. A function $u : (a,b) \to \mathcal{S}$ is continuous at $t \in (a,b)$ if

$$u(t+h) \to u(t) \qquad \text{in } \mathcal{S} \text{ as } h \to 0,$$

and differentiable at $t \in (a,b)$ if there exists a function $v \in \mathcal{S}$ such that

$$\frac{u(t+h) - u(t)}{h} \to v \qquad \text{in } \mathcal{S} \text{ as } h \to 0.$$

The derivative v of u at t is denoted by $u_t(t)$, and if u is differentiable for every $t \in (a,b)$, then $u_t : (a,b) \to \mathcal{S}$ denotes the map $u_t : t \mapsto u_t(t)$.

In other words, u is continuous at t if

$$u(t) = \mathcal{S}\text{-}\lim_{h \to 0} u(t+h),$$

and u is differentiable at t with derivative $u_t(t)$ if

$$u_t(t) = \mathcal{S}\text{-}\lim_{h \to 0} \frac{u(t+h) - u(t)}{h}.$$

We will refer to this derivative as the strong derivative of u if we want to emphasize that it is defined as the limit of difference quotients in $\mathcal{S}$.

The convergence of functions in $\mathcal{S}$ implies uniform pointwise convergence. Thus, if $u(t) = u(\cdot, t)$ is strongly differentiable at t, then the pointwise partial derivative $\partial_t u(x,t)$ exists for every $x \in \mathbb{R}^n$ and $u_t(t) = \partial_t u(\cdot, t) \in \mathcal{S}$.

We define spaces of differentiable Schwartz-valued functions in the natural way. For half-open or closed intervals, we make the obvious modifications to left or right limits at an endpoint.

Definition 5.2. The space $C([a,b]; \mathcal{S})$ consists of the continuous functions $u : [a,b] \to \mathcal{S}$. The space $C^k((a,b); \mathcal{S})$ consists of functions $u : (a,b) \to \mathcal{S}$ that are k-times strongly differentiable in (a,b) with continuous derivatives $\partial_t^j u \in C((a,b); \mathcal{S})$ for $0 \le j \le k$, and $C^\infty((a,b); \mathcal{S})$ is the space of functions with continuous strong derivatives of all orders.

We interpret the initial value problem (5.2) for the heat equation as follows: A solution is a function $u : [0,\infty) \to \mathcal{S}$ that is continuous for $t \ge 0$, so that it makes sense to impose the initial condition at $t = 0$, and continuously differentiable for $t > 0$, so that it makes sense to impose the PDE pointwise in t. That is, for every

$t > 0$, the strong derivative $u_t(t)$ is required to equal $\Delta u(t)$ where $\Delta : \mathcal{S} \to \mathcal{S}$ is the Laplacian operator.

THEOREM 5.3. *If $f \in \mathcal{S}$, there is a unique solution*

$$u \in C([0,\infty);\mathcal{S}) \cap C^1((0,\infty);\mathcal{S}) \tag{5.3}$$

of (5.2). Furthermore, $u \in C^\infty((0,\infty);\mathcal{S})$ and for $t > 0$ it is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t) f(y)\, dy \tag{5.4}$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}. \tag{5.5}$$

PROOF. Since the spatial Fourier transform $\mathcal{F}$ is a continuous linear map on $\mathcal{S}$ with continuous inverse, the time-derivative of u exists if and only if the time derivative of $\hat{u} = \mathcal{F}u$ exists, and

$$\mathcal{F}(u_t) = (\mathcal{F}u)_t\,.$$

Moreover, $u \in C([0,\infty);\mathcal{S})$ if and only if $\hat{u} \in C([0,\infty);\mathcal{S})$, and $u \in C^k((0,\infty);\mathcal{S})$ if and only if $\hat{u} \in C^k((0,\infty);\mathcal{S})$.

Taking the Fourier transform of (5.2) with respect to x, we find that u is a solution if and only if $\hat{u}(k,t)$ satisfies

$$\hat{u}_t = -|k|^2 \hat{u}, \qquad \hat{u}(0) = \hat{f}. \tag{5.6}$$

This ODE has a unique solution $\hat{u} \in C([0,\infty);\mathcal{S}) \cap C^\infty((0,\infty);\mathcal{S})$ given by

$$\hat{u}(k,t) = \hat{f}(k) e^{-t|k|^2}. \tag{5.7}$$

To prove this in detail, suppose first that u satisfies (5.3). Then

$$\hat{u} \in C([0,\infty);\mathcal{S}) \cap C^1((0,\infty);\mathcal{S})\,,$$

which implies that for each fixed $k \in \mathbb{R}^n$ the scalar-valued function $\hat{u}(k,t)$ is pointwise-differentiable with respect to t in $t > 0$ and continuous in $t \geq 0$. Solving the ODE (5.6) with k as a parameter, we find that $\hat{u}$ must be given by (5.7). Conversely, we claim that the function defined by (5.7) is strongly differentiable with derivative

$$\hat{u}_t(k,t) = -|k|^2 \hat{f}(k) e^{-t|k|^2}.$$

To prove this claim, note that for $h > 0$ we have

$$\frac{\hat{u}(k,t+h) - \hat{u}(k,t)}{h} - u_t(k,t) = \hat{f}(k) e^{-t|k|^2} \left(\frac{e^{-h|k|^2} - 1 + h|k|^2}{h} \right).$$

and

$$\frac{e^{-h|k|^2} - 1 + h|k|^2}{h} \to 0 \qquad \text{in } \mathcal{S} \text{ as } h \to 0^+;$$

while for $h < 0$ we have

$$\frac{\hat{u}(k,t+h) - \hat{u}(k,t)}{h} - u_t(k,t) = \hat{f}(k) e^{-(t+h)|k|^2} \left(\frac{1 - h|k|^2 - e^{h|k|^2}}{h} \right),$$

and a similar conclusion follows. Thus, (5.2) has a unique solution that satisfies (5.3). Moreover, using induction, we see that $u \in C^\infty((0,\infty);\mathcal{S})$.

From Example 5.24, we have

$$\mathcal{F}^{-1}\left[e^{-t|k|^2}\right] = \left(\frac{\pi}{t}\right)^{n/2} e^{-|x|^2/4t}.$$

Taking the inverse Fourier transform of (5.7) and using the convolution theorem, we get (5.4)–(5.5). □

This solution of the heat equation satisfies two basic estimates, one in L^2 and the other in L^∞; the L^2 estimate follows from the Fourier representation, and the L^1 estimate follows from the spatial representation. We let $\|\cdot\|_{L^p}$ denote the spatial L^p-norm,

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p \, dx\right)^{1/p}$$

for $1 \le p < \infty$ and the essential supremum for $p = \infty$.

Corollary 5.4. *If $u : [0,\infty) \to \mathcal{S}(\mathbb{R}^n)$ is the solution of (5.2) constructed in Theorem 5.3, then for $t > 0$*

$$\|u(t)\|_{L^2} \le \|f\|_{L^2}, \qquad \|u(t)\|_{L^\infty} \le \frac{1}{(4\pi t)^{n/2}} \|f\|_{L^1}.$$

Proof. By Parseval's inequality and (5.7),

$$\|u(t)\|_{L^2} = (2\pi)^n \|\hat{u}(t)\|_{L^2} \le (2\pi)^n \|\hat{f}\|_{L^2} = \|f\|_{L^2},$$

which gives the first inequality. From (5.4),

$$|u(x,t)| \le \left(\sup_{x\in\mathbb{R}^n} |\Gamma(x,t)|\right) \int_{\mathbb{R}^n} |f(y)| \, dy,$$

and from (5.5)

$$|\Gamma(x,t)| = \frac{1}{(4\pi t)^{n/2}}.$$

The second inequality then follows. □

Using Theorem 5.31, it follows by interpolation between $(p,p') = (2,2)$ and $(p,p') = (\infty,1)$, that for $2 \le p \le \infty$

$$\|u(t)\|_{L^p} \le \frac{1}{(4\pi t)^{n(1/2-1/p)}} \|f\|_{L^{p'}}.$$

The requirement that $u(t) \in \mathcal{S}$ imposes a condition on the behavior of the solution at infinity. A solution of the initial value problem for the heat equation is not unique without the imposition of some kind of growth condition at infinity. A physical interpretation of this nonuniqueness it is that heat can diffuse from infinity into a region of initially zero temperature if the solution grows sufficiently quickly. Mathematically, the nonuniqueness is a consequence of the the fact that the initial condition is imposed on a characteristic surface $t = 0$ of the heat equation, meaning that the heat equation does not determine the second-order normal (time) derivative u_{tt} on $t = 0$ in terms of the second-order tangential (spatial) derivatives u, Du, D^2u.

We cannot solve the heat equation backward in time to obtain a solution $u : [-T, 0] \to \mathcal{S}$ for general final data $f \in \mathcal{S}$, even if $T > 0$ is small. The same argument as the one in the proof of Theorem 5.3 implies that any such solution would be given by (5.7). If, for example, we take $f \in \mathcal{S}$ such that

$$\hat{f}(k) = e^{-\sqrt{1+|k|^2}}$$

then the corresponding solution

$$\hat{u}(k,t) = e^{-t|k|^2 - \sqrt{1+|k|^2}}$$

grows exponentially as $|k| \to \infty$ for every $t < 0$, and therefore $u(t)$ does not belong to $\mathcal{S}$ (or even $\mathcal{S}'$). Physically, this means that the temperature distribution f cannot arise by thermal diffusion from any previous temperature distribution in $\mathcal{S}$ (or, in fact, in $\mathcal{S}'$).

Equivalently, making the time-reversal $t \mapsto -t$, we see that Schwartz-valued solutions of the initial value problem for the backward heat equation

$$u_t = -\Delta u \quad t > 0, \qquad u(x,0) = f(x)$$

need not exist, so that this problem is not well-posed in $\mathcal{S}$. It is possible to obtain a well-posed initial value problem for the backward heat equation by restricting the initial data, for example to a suitable Gevrey space of C^∞-functions whose spatial derivatives decay at a sufficiently fast rate as their order tends to infinity, but these restrictions are typically too strong to be useful in applications.

5.1.2. Sobolev solutions. For any initial data $f \in \mathcal{S}$, the solutions constructed above satisfy an estimate of the form $\|u(t)\|_{L^2} \leq \|f\|_{L^2}$ and we may therefore extend them by continuity and density to arbitrary initial data $f \in L^2$. More generally, similar estimates hold in any Sobolev space H^s (see Section 5.A.8), which allows us to define generalized solutions for $f \in H^s$.

PROPOSITION 5.5. *Suppose that $u : [0,\infty) \to \mathcal{S}$ is the solution of (5.2) constructed in Theorem 5.3. Then for any $s \in \mathbb{R}$*

$$\|u(t)\|_{H^s} \leq \|f\|_{H^s}.$$

PROOF. Using (5.7) and Parseval's identity, we find that

$$\|u(t)\|_{H^s} = (2\pi)^n \left\| \langle k\rangle^s e^{-t|k|^2} \hat{f} \right\|_{L^2} \leq (2\pi)^n \left\| \langle k\rangle^s \hat{f} \right\|_{L^2} = \|f\|_{H^s}.$$

□

For $T > 0$ and $s \in \mathbb{R}$, let $C([0,T]; H^s)$ denote the Banach space of continuous functions $u : [0,T] \to H^s$ equipped with the norm

$$\|u\|_{C([0,T];H^s)} = \sup_{t \in [0,T]} \|u(t)\|_{H^s}.$$

DEFINITION 5.6. Suppose that $T > 0$, $s \in \mathbb{R}$ and $f \in H^s$. A function $u : [0,T] \to H^s$ is a generalized solution of (5.2) if there exists a sequence of solutions $u_n : [0,T] \to \mathcal{S}$ such that $u_n \to u$ in $C([0,T]; H^s)$ as $n \to \infty$.

THEOREM 5.7. *Suppose that $T > 0$, $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R}^n)$. Then there is a unique generalized solution $u \in C([0,T]; H^s)$ of (5.2). The solution is given by*

$$\hat{u}(k,t) = e^{-t|k|^2} \hat{f}(k).$$

Proof. Fix $T > 0$. Since $\mathcal{S}$ is dense in H^s, there is a sequence of functions $f_n \in \mathcal{S}$ such that $f_n \to f$ in H^s. Let $u_n \in C([0,T];\mathcal{S})$ be the solution of (5.2) with initial data f_n. Then, by linearity, $u_n - u_m$ is the solution with initial data $f_n - f_m$, and Proposition 5.5 implies that

$$\sup_{t\in[0,T]} \|u_n(t) - u_m(t)\|_{H^s} \leq \|f_n - f_m\|_{H^s}.$$

Hence, $\{u_n\}$ is a Cauchy sequence in $C([0,T];H^s)$ and therefore there exists a generalized solution $u \in C([0,T];H^s)$ such that $u_n \to u$ as $n \to \infty$.

Suppose that $f, g \in H^s$ and $u, v \in C([0,T];H^s)$ are generalized solutions with $u(0) = f$, $v(0) = g$. If $u_n, v_n \in C([0,T];\mathcal{S})$ are approximate solutions with $u_n(0) = f_n$, $v_n(0) = g_n$, then

$$\begin{aligned}\|u(t) - v(t)\|_{H^s} &\leq \|u(t) - u_n(t)\|_{H^s} + \|u_n(t) - v_n(t)\|_{H^s} + \|v_n(t) - v(t)\|_{H^s}\\ &\leq \|u(t) - u_n(t)\|_{H^s} + \|f_n - g_n\|_{H^s} + \|v_n(t) - v(t)\|_{H^s}\end{aligned}$$

Taking the limit of this inequality as $n \to \infty$, we find that

$$\|u(t) - v(t)\|_{H^s} \leq \|f - g\|_{H^s}.$$

In particular, if $f = g$ then $u = v$, so a generalized solution is unique.

Finally, we have

$$\hat{u}_n(k,t) = e^{-t|k|^2}\hat{f}_n(k).$$

Taking the limit of this expression in $C([0,T];H^s)$, we get the solution for $\hat{u}$. □

Since a unique generalized solution is defined on any time interval $[0,T]$, there is a unique generalized solution $u \in C_{\mathrm{loc}}([0,\infty);H^s)$. We may obtain additional regularity of generalized solutions in time by use of the equation; roughly speaking, we can trade two space-derivatives for one time-derivative.

Proposition 5.8. *If $u \in C([0,T];H^s)$ is a generalized solution of (5.2), then $u \in C^1([0,T];H^{s-2})$ and $u_t = \Delta u$ in $C^1([0,T];H^{s-2})$.*

Proof. Suppose that $u_n \in C([0,T];\mathcal{S})$ and $u_n \to u$ in $C([0,T];H^s)$. Then $u_n \in C^1([0,T];\mathcal{S})$ and $u_{nt} = \Delta u_n$, so $\{u_{nt}\}$ is Cauchy in $C([0,T];H^{s-2})$ since $\{u_n\}$ is Cauchy in H^s and $\Delta : H^s \to H^{s-2}$ is bounded. Hence there exists $v \in C([0,T];H^{s-2})$ such that $u_{nt} \to v$ in $C([0,T];H^{s-2})$. It follows that

$$u \in C([0,T];H^s) \cap C^1([0,T];H^{s-2})$$

with $u_t = v$. Moreover, taking the limit of $u_{nt} = \Delta u_n$ we find that $u_t = \Delta u$ in $C([0,T];H^{s-2})$. □

In contrast with the case of ODEs, the time derivative of the solution lies in a different space than the solution itself: u takes values in H^s, but u_t takes values in H^{s-2}. This feature is typical for PDEs when — as is usually the case — one considers solutions which take values in Banach spaces whose norms depend on only finitely many derivatives. It did not arise for Schwartz-valued solutions, since differentiation is a continuous operation on $\mathcal{S}$.

The above proposition did not use any special properties of the heat equation, and solutions have much greater regularity as a result of the spatially smoothing effect of the evolution; in fact,

$$u \in C([0,\infty);H^s) \cap C^\infty((0,\infty);H^\infty).$$

5.1.3. The heat-equation semigroup. The solution of an $n \times n$ linear first-order system of ODEs for $\vec{u}(t) \in \mathbb{R}^n$,

$$\vec{u}_t = A\vec{u},$$

may be written as

$$\vec{u}(t) = e^{tA}\vec{u}(0) \qquad -\infty < t < \infty$$

where $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix exponential of tA. The solution operators $T(t) = e^{tA}$ form a uniformly continuous one-parameter group. We may consider the heat equation and other linear evolution equations from a similar perspective. There are, however, significant new issues that arise as a result of the fact that the Laplacian and other spatial differential operators are unbounded maps of a Banach space into itself.

Consider the heat equation

$$u_t = \Delta u, \qquad u(x, 0) = f(x)$$

and suppose, for definiteness, that $f \in L^2(\mathbb{R}^n)$. We could equally well consider initial data that lies in other Banach or Hilbert spaces, such as L^1 or H^s. From Theorem 5.7, with $s = 0$, there is a unique generalized solution $u : [0, \infty) \to L^2$ of the heat equation. For each $t \geq 0$ we may therefore define a bounded linear map $T(t) : L^2 \to L^2$ by $T(t) : f \mapsto u(t)$. Thus, $T(t)$ is the flow or solution operator for the heat equation that maps the initial data at time 0 to the solution at time t. In particular, $T(0) = I$ is the identity.

Since the PDE does not depend explicitly on time, we have

$$T(s + t) = T(s)T(t) \qquad \text{for all } s, t \geq 0, \tag{5.8}$$

so the operators $\{T(t) : t \geq 0\}$ form a one-parameter semigroup. They do not form a group because $T(-t)$ is undefined for $t < 0$ and the operators $T(t)$ are not invertible. This irreversibilty does not arise in the case of ODEs.

The semigroup property in (5.8) is obvious from the explicit Fourier representation (5.7) since

$$e^{-(s+t)|k|^2} = e^{-s|k|^2} e^{-t|k|^2}.$$

It is less obvious from the spatial representation (5.4), but follows from the fact that

$$\Gamma^{s+t} = \Gamma^s * \Gamma^t$$

where the $*$ denotes the spatial convolution and $\Gamma^t(x) = \Gamma(x, t)$.

This semigroup is strongly continuous, meaning that for each $f \in L^2$, the map $t \mapsto T(t)f$ from $[0, \infty)$ into L^2 is continuous; equivalently $T(t+h) \to T(t)$ as $h \to 0$ (or $h \to 0^+$ if $t = 0$) with respect to the strong operator topology. It is *not* true, however, that $T(t + h) \to T(t)$ as $h \to 0$ uniformly with respect to the operator norm, as is the case for ODEs.

We also use the notation

$$T(t) = e^{t\Delta}$$

and interpret $T(t)$ as the operator exponential of $t\Delta$. Equation (5.8) then becomes the usual exponential formula

$$e^{(s+t)\Delta} = e^{s\Delta} e^{t\Delta}.$$

It is remarkable that although the Laplacian is an unbounded linear operator

$$\Delta : H^2 \subset L^2 \to L^2$$

on L^2, the forward-in-time solution operators $T(t) = e^{t\Delta}$ that it generates are bounded.

In this discussion, we began with the heat equation and the Laplacian and derived the corresponding semigroup. We can instead begin with a semigroup and determine the operator that generates it. A key question is then to characterize the operators that generate a semigroup. We will briefly describe some basic results of semigroup theory without proof. For a detailed discussion see, for example, [**4**].

DEFINITION 5.9. Let X be a Banach space. A one-parameter, strongly continuous (or C_0) semigroup on X is a family $\{T(t) : t \geq 0\}$ of bounded linear operators $T(t) : X \to X$ such that

(1) $T(0) = I$;
(2) $T(s)T(t) = T(s+t)$ for all $s, t \geq 0$;
(3) For every $f \in X$, $T(t)f \to f$ strongly in X as $t \to 0^+$.

The semigroup is said to be a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$, where $\|\cdot\|$ denotes the operator norm.

Explicitly, (3) means that

$$\|T(t)f - f\|_X \to 0 \qquad \text{as } t \to 0^+.$$

If this condition holds, then the semigroup property implies that $T(t+h)f \to T(t)f$ in X as $h \to 0$ for every $t > 0$, not only for $t = 0$.

The heat equation semigroup on $X = L^2(\mathbb{R}^n)$ is an example of a contraction semigroup. The term 'contraction' is not used here in a strict sense. The wave equation and Schrödinger equation also generate contraction semigroups (and, in fact, groups since their evolution is time-reversible). Thus, the norm of the solution of a contraction semigroup is not required to be strictly decreasing in time and it may, for example, remain constant.

DEFINITION 5.10. Suppose that $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup on a Banach space X. The generator A of the semigroup is the linear operator in X with domain $\mathcal{D}(A)$,

$$A : \mathcal{D}(A) \subset X \to X,$$

defined as follows:

(1) $f \in \mathcal{D}(A)$ if and only if the limit
$$\lim_{h \to 0^+} \frac{T(h)f - f}{h}$$
exists with respect to the strong (norm) topology of X;
(2) if $f \in \mathcal{D}(A)$, then
$$Af = \lim_{h \to 0^+} \frac{T(h)f - f}{h}.$$

DEFINITION 5.11. An operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is closed if whenever $\{f_n\}$ is a sequence of points in $\mathcal{D}(A)$ such that $f_n \to f$ and $Af_n \to g$ in X as $n \to \infty$, then $f \in \mathcal{D}(A)$ and $Af = g$.

A bounded operator with dense domain $\mathcal{D}(A)$ is closed if and only if $\mathcal{D}(A) = X$ are closed. Differential operators defined in terms of weak derivatives give typical examples of unbounded closed operators.

THEOREM 5.12. *If A is the generator of a strongly continuous semigroup $\{T(t)\}$ on a Banach space X, then A is closed and its domain $\mathcal{D}(A)$ is dense in X.*

The semigroup $T(t)$ may be recovered from its generator in various ways, many of which generalize ways of defining the standard exponential function in a manner that is appropriate for an operator that is unbounded.

Finally, we state some conditions for an operator to generate a semigroup.

DEFINITION 5.13. Suppose that $A : \mathcal{D}(A) \subset X \to X$ is a closed linear operator in a Banach space X and $\mathcal{D}(A)$ is dense in X. A complex number $\lambda \in \mathbb{C}$ is in the resolvent set of A if $\lambda I - A : \mathcal{D}(A) \subset X \to X$ is one-to-one and onto and with bounded inverse

$$R(\lambda, A) = (\lambda I - A)^{-1} : X \to X. \tag{5.9}$$

called the resolvent of A.

The Hille-Yoshida theorem, provides a necessary and sufficient condition for an operator A to generate a strongly continuous semigroup

THEOREM 5.14. *A linear operator $A : \mathcal{D}(A) \subset X \to X$ is the generator of a strongly continuous semigroup $\{T(t); t \geq 0\}$ in X if and only if there exist constants $M \geq 1$ and $a \in \mathbb{R}$ such that the following conditions are satisfied:*

1. *the domain $\mathcal{D}(A)$ is dense in X and A is closed;*
2. *every $\lambda \in \mathbb{R}$ such that $\lambda > a$ belongs to the resolvent set of A;*
3. *if $\lambda > a$ and $n \in \mathbb{N}$, then*
$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - a)^n}$$
where the resolvent $R(\lambda, A)$ is defined in (5.9).

In that case,
$$\|T(t)\| \leq Me^{at} \qquad \text{for all } t \geq 0.$$

The Lummer-Phillips theorem provides a more easily checked condition (that A is 'm-dissipative') for A to generate a contraction semigroup on a Hilbert space.

Appendix

5.A. The Schwartz space and the Fourier transform

May the Schwartz be with you![1]

In this section, we summarize some results about Schwartz functions, tempered distributions, and the Fourier transform. For complete proofs, see [**13, 15**].

5.A.1. The Schwartz space. Since we will study the Fourier transform, we consider complex-valued functions here.

Definition 5.15. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and

$$x^\alpha \partial^\beta f(x) \to 0 \qquad \text{as } |x| \to \infty$$

for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. For $\alpha, \beta \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$ let

$$\|f\|_{\alpha,\beta} = \sup_{\mathbb{R}^n} \left| x^\alpha \partial^\beta f \right|. \tag{5.10}$$

A sequence of functions $\{f_k : k \in \mathbb{N}\}$ converges to a function f in $\mathcal{S}(\mathbb{R}^n)$ if

$$\|f_n - f\|_{\alpha,\beta} \to 0 \qquad \text{as } k \to \infty$$

for every $\alpha, \beta \in \mathbb{N}_0^n$.

That is, the Schwartz space consists of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power; we say, for short, that Schwartz functions are rapidly decreasing. When there is no ambiguity, we will write $\mathcal{S}(\mathbb{R}^n)$ as $\mathcal{S}$.

Example 5.16. The function $f(x) = e^{-|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$. More generally, if p is any polynomial, then $g(x) = p(x)\, e^{-|x|^2}$ belongs to $\mathcal{S}$.

Example 5.17. The function

$$f(x) = \frac{1}{(1 + |x|^2)^k}$$

does not belongs to $\mathcal{S}$ for any $k \in \mathbb{N}$ since $|x|^{2k} f(x)$ does not decay to zero as $|x| \to \infty$.

Example 5.18. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-x^2} \sin\left(e^{x^2}\right)$$

does not belong to $\mathcal{S}(\mathbb{R})$ since $f'(x)$ does not decay to zero as $|x| \to \infty$.

The space $\mathcal{D}(\mathbb{R}^n)$ of smooth complex-valued functions with compact support is contained in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. If $f_k \to f$ in $\mathcal{D}$ (in the sense of Definition 3.7), then $f_k \to f$ in $\mathcal{S}$, so $\mathcal{D}$ is continuously imbedded in $\mathcal{S}$. Furthermore, if $f \in \mathcal{S}$, and $\eta \in C_c^\infty(\mathbb{R}^n)$ is a cutoff function with $\eta_k(x) = \eta(x/k)$, then $\eta_k f \to f$ in $\mathcal{S}$ as $k \to \infty$, so $\mathcal{D}$ is dense in $\mathcal{S}$.

[1] Spaceballs

The topology of $\mathcal{S}$ is defined by the countable family of semi-norms $\|\cdot\|_{\alpha,\beta}$ given in (5.10). This topology is not derived from a norm, but it is metrizable; for example, we can use as a metric

$$d(f,g) = \sum_{\alpha,\beta\in\mathbb{N}_0^n} \frac{c_{\alpha,\beta}\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}$$

where the $c_{\alpha,\beta} > 0$ are any positive constants such that $\sum_{\alpha,\beta\in\mathbb{N}_0^n} c_{\alpha,\beta}$ converges. Moreover, $\mathcal{S}$ is complete with respect to this metric. A complete, metrizable topological vector space whose topology may be defined by a countable family of semi-norms is called a Fréchet space. Thus, $\mathcal{S}$ is a Fréchet space.

If we want to make explicit that a limit exists with respect to the Schwartz topology, we write

$$f = \mathcal{S}\text{-}\lim_{k\to\infty} f_k,$$

and call f the $\mathcal{S}$-limit of $\{f_k\}$.

If $f_k \to f$ as $k \to \infty$ in $\mathcal{S}$, then $\partial^\alpha f_k \to \partial^\alpha f$ for any multi-index $\alpha \in \mathbb{N}_0^n$. Thus, the differentiation operator $\partial^\alpha : \mathcal{S} \to \mathcal{S}$ is a continuous linear map on $\mathcal{S}$.

5.A.2. Tempered distributions. Tempered distributions are distributions (*c.f.* Section 3.3) that act continuously on Schwartz functions. Roughly speaking, we can think of tempered distributions as distributions that grow no faster than a polynomial at infinity.[2]

Definition 5.19. A tempered distribution T on $\mathbb{R}^n$ is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The topological vector space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{S}'$. If $\langle T, f\rangle$ denotes the value of $T \in \mathcal{S}'$ acting on $f \in \mathcal{S}$, then a sequence $\{T_k\}$ converges to T in $\mathcal{S}'$, written $T_k \rightharpoonup T$, if

$$\langle T_k, f\rangle \to \langle T, f\rangle \qquad \text{for every } f \in \mathcal{S}.$$

Since $\mathcal{D} \subset \mathcal{S}$ is densely and continuously imbedded, we have $\mathcal{S}' \subset \mathcal{D}'$. Moreover, a distribution $T \in \mathcal{D}'$ extends uniquely to a tempered distribution $T \in \mathcal{S}'$ if and only if it is continuous on $\mathcal{D}$ with respect to the topology on $\mathcal{S}$.

Every function $f \in L^1_{\text{loc}}$ defines a regular distribution $T_f \in \mathcal{D}'$ by

$$\langle T_f, \phi\rangle = \int f\phi\, dx \qquad \text{for all } \phi \in \mathcal{D}.$$

If $|f| \le p$ is bounded by some polynomial p, then T_f extends to a tempered distribution $T_f \in \mathcal{S}'$, but this is not the case for functions f that grow too rapidly at infinity.

Example 5.20. The locally integrable function $f(x) = e^{|x|^2}$ defines a regular distribution $T_f \in \mathcal{D}'$ but this distribution does not extend to a tempered distribution.

Example 5.21. If $f(x) = e^x \cos(e^x)$, then $T_f \in \mathcal{D}'(\mathbb{R})$ extends to a tempered distribution even though the values of $f(x)$ grow exponentially as $x \to \infty$. This

[2]The name 'tempered distribution' is short for 'distribution of temperate growth,' meaning growth that is at most polynomial.

tempered distribution is the distributional derivative $T_f = (T_g)'$ of the regular distribution T_g where $f = g'$ and $g(x) = \sin(e^x)$:

$$\langle f, \phi\rangle = -\langle g, \phi'\rangle = -\int \sin(e^x)\phi(x)\,dx \qquad \text{for all } \phi \in \mathcal{S}.$$

The distribution T_f is decreasing in a weak sense at infinity because of the rapid oscillations of f.

EXAMPLE 5.22. The series

$$\sum_{n\in\mathbb{N}} \delta^{(n)}(x-n)$$

where $\delta^{(n)}$ is the nth derivative of the δ-function converges to a distribution in $\mathcal{D}'(\mathbb{R})$, but it does not converge in $\mathcal{S}'(\mathbb{R})$ or define a tempered distribution.

We define the derivative of tempered distributions in the same way as for distributions. If $\alpha \in \mathbb{N}_0^n$ is a multi-index, then

$$\langle \partial^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha \phi\rangle.$$

We say that a C^∞-function f is slowly growing if the function and all of its derivatives are of polynomial growth, meaning that for every $\alpha \in \mathbb{N}_0^n$ there exists a constant C_α and an integer N_α such that

$$|\partial^\alpha f(x)| \le C_\alpha \left(1 + |x|^2\right)^{N_\alpha}.$$

If f is C^∞ and slowly growing, then $f\phi \in \mathcal{S}$ whenever $\phi \in \mathcal{S}$, and multiplication by f is a continuous map on $\mathcal{S}$. Thus for $T \in \mathcal{S}'$, we may define the product $fT \in \mathcal{S}'$ by

$$\langle fT, \phi\rangle = \langle T, f\phi\rangle.$$

5.A.3. The Fourier transform on $\mathcal{S}$. The Schwartz space is a natural one to use for the Fourier transform. Differentiation and multiplication exchange rôles under the Fourier transform and therefore so do the properties of smoothness and rapid decrease. As a result, the Fourier transform is an automorphism of the Schwartz space. By duality, the Fourier transform is also an automorphism of the space of tempered distributions.

DEFINITION 5.23. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^n}\int f(x)e^{-ik\cdot x}\,dx. \tag{5.11}$$

The inverse Fourier transform of f is the function $\check{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\check{f}(x) = \int f(k)e^{ik\cdot x}\,dk.$$

We generally use x to denote the variable on which a function f depends and k to denote the variable on which its Fourier transform depends.

EXAMPLE 5.24. For $\sigma > 0$, the Fourier transform of the Gaussian

$$f(x) = \frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-|x|^2/2\sigma^2}$$

is the Gaussian

$$\hat{f}(k) = \frac{1}{(2\pi)^n} e^{-\sigma^2|k|^2/2}$$

The Fourier transform maps differentiation to multiplication by a monomial and multiplication by a monomial to differentiation. As a result, $f \in \mathcal{S}$ if and only if $\hat{f} \in \mathcal{S}$, and $f_n \to f$ in $\mathcal{S}$ if and only if $\hat{f}_n \to \hat{f}$ in $\mathcal{S}$.

THEOREM 5.25. *The Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ defined by $\mathcal{F} : f \mapsto \hat{f}$ is a continuous, one-to-one map of $\mathcal{S}$ onto itself. The inverse $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$ is given by $\mathcal{F}^{-1} : f \mapsto \check{f}$. If $f \in \mathcal{S}$, then*

$$\mathcal{F}\left[\partial^\alpha f\right] = (ik)^\alpha \hat{f}, \qquad \mathcal{F}\left[(-ix)^\beta f\right] = \partial^\beta \hat{f}.$$

The Fourier transform maps the convolution product of two functions to the pointwise product of their transforms.

THEOREM 5.26. *If $f, g \in \mathcal{S}$, then the convolution $h = f * g \in \mathcal{S}$, and*

$$\hat{h} = (2\pi)^n \hat{f}\hat{g}.$$

If $f, g \in \mathcal{S}$, then

$$\int f\overline{g}\, dx = (2\pi)^n \int \hat{f}\overline{\hat{g}}\, dk.$$

In particular,

$$\int |f|^2\, dx = (2\pi)^n \int |\hat{f}|^2\, dk.$$

5.A.4. The Fourier transform on $\mathcal{S}'$. The main reason to introduce tempered distributions is that their Fourier transform is also a tempered distribution. If $\phi, \psi \in \mathcal{S}$, then by Fubini's theorem

$$\begin{aligned}
\int \phi\hat{\psi}\, dx &= \int \phi(x) \left[\frac{1}{(2\pi)^n} \int \psi(y) e^{-ix\cdot y}\, dy\right] dx \\
&= \int \left[\frac{1}{(2\pi)^n} \int \phi(x) e^{-ix\cdot y}\, dx\right] \psi(y)\, dy \\
&= \int \hat{\phi}\psi\, dx.
\end{aligned}$$

This motivates the following definition for the Fourier transform of a tempered distribution which is compatible with the one for Schwartz functions.

DEFINITION 5.27. If $T \in \mathcal{S}'$, then the Fourier transform $\hat{T} \in \mathcal{S}'$ is the distribution defined by

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle \qquad \text{for all } \phi \in \mathcal{S}.$$

The inverse Fourier transform $\check{T} \in \mathcal{S}'$ is the distribution defined by

$$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle \qquad \text{for all } \phi \in \mathcal{S}.$$

We also write $\hat{T} = \mathcal{F}T$ and $\check{T} = \mathcal{F}^{-1}T$. The linearity and continuity of the Fourier transform on $\mathcal{S}$ implies that $\hat{T}$ is a linear, continuous map on $\mathcal{S}$, so the Fourier transform of a tempered distribution is a tempered distribution. The invertibility of the Fourier transform on $\mathcal{S}$ implies that $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is invertible with inverse $\mathcal{F}^{-1} : \mathcal{S}' \to \mathcal{S}'$.

EXAMPLE 5.28. If δ is the delta-function supported at 0, $\langle \delta, \phi \rangle = \phi(0)$, then

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^n} \int \phi(x)\, dx = \left\langle \frac{1}{(2\pi)^n}, \phi \right\rangle.$$

Thus, the Fourier transform of the δ-function is the constant function $(2\pi)^{-n}$. This result is consistent with Example 5.24. We have for the Gaussian δ-sequence that

$$\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-|x|^2/2\sigma^2} \rightharpoonup \delta \qquad \text{in } \mathcal{S}' \text{ as } \sigma \to 0.$$

The corresponding Fourier transform of this limit is

$$\frac{1}{(2\pi)^n} e^{-\sigma^2|k|^2/2} \rightharpoonup \frac{1}{(2\pi)^n} \qquad \text{in } \mathcal{S}' \text{ as } \sigma \to 0.$$

If $T \in \mathcal{S}'$, it follows directly from the definitions and the properties of Schwartz functions that

$$\langle \widehat{\partial^\alpha T}, \phi \rangle = \langle \partial^\alpha T, \hat{\phi} \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \hat{\phi} \rangle = \langle T, \widehat{(ik)^\alpha \phi} \rangle = \langle \hat{T}, (ik)^\alpha \phi \rangle = \langle (ik)^\alpha \hat{T}, \phi \rangle,$$

with a similar result for the inverse transform. Thus,

$$\widehat{\partial^\alpha T} = (ik)^\alpha \hat{T}, \qquad \widehat{(-ix)^\beta T} = \partial^\beta \hat{T}.$$

The Fourier transform does not define a map of the test function space $\mathcal{D}$ into itself, since the Fourier transform of a compactly supported function does not, in general, have compact support. Thus, the Fourier transform of a distribution $T \in \mathcal{D}'$ is not, in general, a distribution $\hat{T} \in \mathcal{D}'$; this explains why we define the Fourier transform for the smaller class of tempered distributions.

The Fourier transform maps the space $\mathcal{D}$ onto a space $\mathcal{Z}$ of real-analytic functions,[3] and one can define the Fourier transform of a general distribution $T \in \mathcal{D}'$ as an ultradistribution $\hat{T} \in \mathcal{Z}'$ acting on $\mathcal{Z}$. We will not consider this theory further here.

5.A.5. The Fourier transform on L^1. If $f \in L^1(\mathbb{R}^n)$, then

$$\left| \int f(x) e^{-ik\cdot x}\, dx \right| \leq \int |f|\, dx,$$

so we may define the Fourier transform $\hat{f}$ directly by the absolutely convergent integral in (5.11). Moreover,

$$\left| \hat{f}(k) \right| \leq \frac{1}{(2\pi)^n} \int |f|\, dx.$$

It follows by approximation of f by Schwartz functions that $\hat{f}$ is a uniform limit of Schwartz functions, and therefore $\hat{f} \in C_0$ is a continuous function that approaches zero at infinity. We therefore get the following Riemann-Lebesgue lemma.

[3]A function $\phi : \mathbb{R} \to \mathbb{C}$ belongs to $\mathcal{Z}(\mathbb{R})$ if and only if it extends to an entire function $\phi : \mathbb{C} \to \mathbb{C}$ with the property that, writing $z = x + iy$, there exists $a > 0$ and for each $k = 0, 1, 2, \dots$ a constant C_k such that

$$\left| z^k \phi(z) \right| \leq C_k e^{a|y|}.$$

THEOREM 5.29. *The Fourier transform is a bounded linear map* $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ *and*

$$\left\|\hat{f}\right\|_{L^\infty} \leq \frac{1}{(2\pi)^n} \|f\|_{L^1}.$$

The range of the Fourier transform on L^1 is not all of C_0, however, and it is difficult to characterize.

5.A.6. The Fourier transform on L^2. The next theorem, called Parseval's theorem, states that the Fourier transform preserves the L^2-inner product and norm, up to factors of 2π. It follows that we may extend the Fourier transform by density and continuity from $\mathcal{S}$ to an isomorphism on L^2 with the same properties. Explicitly, if $f \in L^2$, we choose any sequence of functions $f_k \in \mathcal{S}$ such that f_k converges to f in L^2 as $k \to \infty$. Then we define $\hat{f}$ to be the L^2-limit of the $\hat{f}_k$. Note that it is necessary to use a somewhat indirect approach to define the Fourier transform on L^2, since the Fourier integral in (5.11) does not converge if $f \in L^2 \setminus L^1$.

THEOREM 5.30. *The Fourier transform* $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a one-to-one, onto bounded linear map. If* $f, g \in L^2(\mathbb{R}^n)$*, then*

$$\int f\overline{g}\, dx = (2\pi)^n \int \hat{f}\overline{\hat{g}}\, dk.$$

In particular,

$$\int |f|^2 \, dx = (2\pi)^n \int |\hat{f}|^2 \, dk.$$

5.A.7. The Fourier transform on L^p. The boundedness of the Fourier transform $\mathcal{F} : L^p \to L^{p'}$ for $1 < p < 2$ follows from its boundedness for $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{F} : L^2 \to L^2$ by use of the following Riesz-Thorin interpolation theorem.

THEOREM 5.31. *Let* Ω *be a measure space and* $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$. *Suppose that*

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \to L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

is a linear map such that $T : L^{p_i}(\Omega) \to L^{q_i}(\Omega)$ *for* $i = 0, 1$ *and*

$$\|Tf\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \qquad \|Tf\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}$$

for some constants M_0, M_1. *If* $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then $T : L^p(\Omega) \to L^q(\Omega)$ *maps* $L^p(\Omega)$ *into* $L^q(\Omega)$ *and*

$$\|Tf\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}$$

In this theorem, $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ denotes the vector space of all complex-valued functions of the form $f = f_0 + f_1$ where $f_0 \in L^{p_0}(\Omega)$ and $f_1 \in L^{p_1}(\Omega)$.

An immediate consequence of this theorem and the L^1-L^2 estimates for the Fourier transform is the following Hausdorff-Young theorem.

THEOREM 5.32. *Suppose that* $1 \leq p \leq 2$. *The Fourier transform is a bounded linear map* $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ *and*

$$\|\mathcal{F}f\|_{L^{p'}} \leq \frac{1}{(2\pi)^n} \|f\|_{L^p}.$$

If $1 \le p < 2$, the range of the Fourier transform on L^p is not all of $L^{p'}$, and there exist functions $f \in L^{p'}$ whose inverse Fourier transform is a tempered distribution that is not regular. Correspondingly, if $p > 2$ the range of $\mathcal{F} : L^p \to \mathcal{S}'$ contains non-regular distributions. For example, $1 \in L^\infty$ and $\mathcal{F}(1) = \delta$.

5.A.8. The Sobolev spaces $H^s(\mathbb{R}^n)$. A function belongs to L^2 if and only if its Fourier transform belongs to L^2 and the Fourier transform preserves the L^2-norm. As a result, the Fourier transform provides a simple way to define L^2-Sobolev spaces on $\mathbb{R}^n$, including ones of fractional and negative order. This approach does not generalize to L^p-Sobolev spaces with $p \neq 2$, since it is not easy to characterize when a function belongs to L^p in terms of its Fourier transform.

We define a function $\langle\cdot\rangle : \mathbb{R}^n \to \mathbb{R}$ by

$$\langle x \rangle = \left(1 + |x|^2\right)^{1/2}.$$

This function grows linearly at infinity, like $|x|$, but is bounded away from zero. (There should be no confusion with the use of angular brackets to denote a duality pairing.)

Definition 5.33. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ is a regular distribution such that

$$\int \langle k \rangle^{2s} \left|\hat{f}(k)\right|^2 \, dk < \infty.$$

The inner product and norm of $f, g \in H^s$ are defined by

$$(f,g)_{H^s} = \int \langle k \rangle^{2s} \overline{\hat{f}(k)} \hat{g}(k)\, dk, \qquad \|f\|_{H^s} = \left(\int \langle k \rangle^{2s} \left|\hat{f}(k)\right|^2 \, dk\right)^{1/2}.$$

These Sobolev spaces form a decreasing scale of Hilbert spaces with H^s continuously imbedded in H^r for $s > r$.

We may give a spatial description of H^s in terms of the operator $\Lambda : \mathcal{S}' \to \mathcal{S}'$ with symbol $\langle k \rangle$ defined by

$$\Lambda = (I - \Delta)^{1/2}, \qquad \widehat{(\Lambda f)}(k) = \langle k \rangle \hat{f}(k).$$

Then $f \in H^s$ if and only if $\Lambda^s f \in L^2$. Roughly speaking, $f \in H^s$ if f has s weak derivatives (or integrals if $s < 0$) that belong to L^2.

Example 5.34. If $\delta \in \mathcal{S}'(\mathbb{R}^n)$, then $\hat{\delta} = (2\pi)^{-n}$ and

$$\int \langle k \rangle^{2s} \hat{\delta}^2 \, dk = \frac{1}{(2\pi)^{2n}} \int \langle k \rangle^{2s} \, dk$$

converges if $2s < -n$. Thus, $\delta \in H^s(\mathbb{R}^n)$ if $s < -n/2$. More generally, every compactly supported distribution belongs to H^s for some $s \in \mathbb{R}$.

Example 5.35. The Fourier transform of $1 \in \mathcal{S}'$, given by $\hat{1} = \delta$, is not a regular distribution. Thus, $1 \notin H^s$ for any $s \in \mathbb{R}$.

We let

$$H^\infty = \bigcap_{s \in \mathbb{R}} H^s, \qquad H^{-\infty} = \bigcup_{s \in \mathbb{R}} H^s.$$

Then $\mathcal{S} \subset H^\infty \subset H^{-\infty} \subset \mathcal{S}'$ and by the Sobolev imbedding theorem $H^\infty \subset C^\infty$.

Bibliography

[1] R. A. Adams and J. Fournier, *Sobolev Spaces*, Elsevier, 2003.
[2] S. Alinhac and P. Gerard, *Pseudo differential operators and Nash Moser*, Amer. Math. Soc.
[3] J. Duoandikoetxea, *Fourier Analysis*, Amer. Math. Soc., Providence, RI, 1995.
[4] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.
[5] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1993.
[6] L. C. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.
[7] G. Folland, *Real Analysis: Modern Techniques and Applications*, Wiley, New York, 1984.
[8] R. Gariepy and W. Ziemer, *Modern Real Analysis*, PWS Publishing, Boston, 1995.
[9] D. J. H. Garling, *Inequalities*, Cambridge University Press, 2007.
[10] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.
[11] F. Jones, *Lebesgue Integration on Euclidean Space*, Jones & Bartlett, Boston, 1993.
[12] Q. Han and F. Lin, *Elliptic Partial Differential Equations*, Courant Lecture Notes in Mathematics, Vol. 1, AMS, New York, 1997.
[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd ed., Springer-Verlag, 1990.
[14] G. Leoni, *A First Course in Sobolev Spaces*, Amer. Math. Soc., Providence, RI, 2009.
[15] J. Rauch, *Partial Differential Equations*, Springer-Verlag, 1991.
[16] S. Shkoller, *Notes on L^p and Sobolev Spaces.*
[17] M. E. Taylor, *Measure Theory and Integration*, Amer. Math. Soc., Providence, RI, 2006.
[18] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, 1980.
[19] W. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.