Problem set 1

Math 218A, Fall 2009

1. Suppose that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are multi-indices. We say that $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for $1 \le i \le n$, and define

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!}.$$

Prove the generalized Leibniz rule

$$\partial^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} u \, \partial^\beta v.$$

2. A function $u \in C^2(\Omega)$ is *subharmonic* in a domain Ω if $-\Delta u \le 0$, and *superharmonic* if $-\Delta u \ge 0$. Prove that if a subharmonic function attains its maximum at any point of Ω, then it is a constant function. Suppose that

$$u, v, w \in C^2(\Omega) \cap C(\overline{\Omega}),$$

where u is subharmonic, v is harmonic, and w is superharmonic. If $u \le v \le w$ on $\partial\Omega$, prove that $u \le v \le w$ on $\overline{\Omega}$.

3. Find the Green's function for Laplace's equation in the unit ball in $\mathbb{R}^n$, and write out the Green's function representation of the solution of

$$\begin{aligned} -\Delta u = f \qquad |x| < 1, \\ u = 0 \qquad |x| = 1. \end{aligned}$$

Hint. Method of images.

4. Suppose that $\{u_n\}$ is a decreasing sequence of harmonic functions defined in a domain Ω such that $\{u_n(x)\}$ converges for some point $x \in \Omega$. Prove that the sequence converges uniformly on every closed, bounded subset of Ω, and that the limit is harmonic. Hint. Harnack's inequality.