Problem set 2
Math 218A, Fall 2009

1. Suppose that Ω is a connected, bounded open set with C^1-boundary that satisfies the interior sphere condition at every point of its boundary. Use a maximum principle argument to prove that a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Neumann problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

is unique up to an arbitrary additive constant.

2. Suppose that Ω is a bounded open set and $0 < \alpha \leq 1$. If $u : \Omega \to \mathbb{R}$, let

$$|u|_0 = \sup_\Omega |u|, \qquad |u|_1 = \sup_\Omega |Du|,$$
$$[u]_{1,\alpha} = \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x - y|^\alpha}.$$

Prove the following interpolation inequality: For any $\epsilon > 0$ there exists a constant C_ϵ such that

$$|u|_1 \leq C_\epsilon |u|_0 + \epsilon [u]_{1,\alpha} \qquad \text{for all } u \in C^{1,\alpha}(\overline{\Omega}).$$

Hint. Assume the inequality is false and use a compactness argument to derive a contradiction.

3. Suppose that $f \in C_c^\infty(\mathbb{R}^n)$ and let $u = \Gamma * f$ be the Newtonian potential of f. If $\operatorname{spt} f \subset B_R(0)$, prove that

$$\sup_{B_R(0)} (|u| + |Du|) \leq CR^2 \sup |f|$$

where C is a constant depending only on n.

4. Give an example of a function $f \in C_c(\mathbb{R}^2)$ such that there is no solution $u \in C^2(\mathbb{R}^2)$ of Poisson's equation

$$u_{xx} + u_{yy} = f(x, y).$$